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Grade 7/8 Math Circles

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Estimation

Introduction

If you ever find yourself without a calculator, a ruler, a computer, or if you are missing a few pieces of key information, solving math problems can become increasingly difficult. Often times, if you are in one of these situations, one of your options is to simply *guess* the answer.

Guessing can be effective at times, but an even more power tool which can be used in these situations is *estimation*!

Estimations can be done in just about any way that you want, as long as you can justify your reasoning. A few different ways you can estimate things, or different ways in which you can define an estimation are:

- A rough calculation.
- An educated guess.
- A combined sequence of assumptions.

The first thing that I want everyone to do today is *guess* how many of your handouts do think can fit into the box at the front of the room? Remember that a guess is different than an educated guess, or an estimation. Simply write down your initial thought without doing any kind of calculation.

Write your guess in the space below. Later into the class, we will try *estimating* how many handouts can fit into the box and see if we can get closer to the actual answer!

I guess that ____________________ of these handouts can fit into the box at the front of the classroom.

Estimation Techniques

The most common use for estimating is doing calculations without the help of a calculator, or even a pencil and paper.

Some of you might be comfortable with doing mental math already, but today we'll try doing some calculations that are almost *impossible* to do purely in your head. To solve questions like these, the best we can do is an estimation, to give us a rough idea of what the answer should actually be!

Today we'll try a few estimation techniques to do with:

- Finding the values of large exponents.
- Calculating areas.
- Estimating how much $1,000 of quarters would weigh.

Example 1:

Using purely mental math, quickly estimate the value of each of the calculations below.

(a) 248×102

(b) $1,584 \div 32$

(c) $\sqrt{957}$

Quickly check by using an estimation, which of these values *must* be *incorrect*.

(a) $3,575 \div 625 = 6.5$

(b) $\sqrt{\pi} \approx 1.772$

(c) $\frac{72}{15} \times \frac{21}{19} = 4.287$

Exponent Estimations

Before we get into an example of an estimation dealing with exponents, we first have to look in depth at exactly how exponents work. Consider doing the following calculation with any non-zero number 'a':

$$a^3 \times a^2 = (a \times a \times a) \times (a \times a) = a^5$$

Another way of looking at this calculation, is to simply *add* the values of the exponents together.

$$a^3 \times a^2 = a^{3+2} = a^5$$

Similarly to how we can *add* exponents together (if they have the same base number), we can also split an exponent into smaller pieces if we choose to do so.

$$a^7 = a^2 \times a^3 \times a^2$$

As a matter of fact, we can split up our exponents however we like! We can even split them into fractions, decimals or even negative numbers.

$$a^1 = a^{\frac{1}{2}} \times a^{\frac{1}{2}}$$

Judging from the equation above, $a^{\frac{1}{2}}$ must be the exact same thing as _______________!

There is one more exponent trick that we will look at today. Consider the following:

$$(a^2)^3 = a^2 \times a^2 \times a^2 = a^6$$

Another way of looking at this calculation, is to simply *multiply* the values of the exponents together.

$$(a^2)^3 = a^{2\times 3} = a^6$$

To summarize our results in general from the last page, we discovered these three properties:

- $a^b \times a^c = a^{b+c}$
- $a^{\frac{1}{2}} = \sqrt{a}$
- $(a^b)^c = a^{b \times c}$

Example 2:

Use the facts about exponents above to help you estimate the following:

(a) π^6

(b) π^{20}

(c) 2^{20}

Scientific Notation

For questions like the ones in example 2, you end up getting numbers that are quite large. In fact, sometimes the numbers get so large, that even writing them down can take a *very* long time.

A 'googol' for example is an *extremely* large number. A googol starts with the number 1, and then has 100 zeros following it!

$$1\,Googol = 1000000000000000000000000000000000000000000000000000000000...$$

The entire number can't even fit on the page! To fix this problem, we use something known as scientific notation.

What scientific notation does is take out as many powers of 10 from a number as possible, until you are left with a number between 1 and 10. In the case of a googol, there are 100 zeros, so that must mean there must be 100 powers of 10. Based on this, we can re-write a googol as:

$$1\,Googol = 1 \times 10^{100}$$

Here are a few examples of different numbers being written in scientific notation:

- $4,000,000 = 4 \times 10^6$
- $321 = 3.21 \times 10^2$
- $505,000,000,000 = 5.05 \times 10^{11}$

Example 3:

Try writing the following numbers in scientific notation:

(a) 7,250,000,000

(b) 1,800

Try the following calculations in scientific notation:

(a) $(5 \times 10^5) \times (4 \times 10^3)$

(b) $(8 \times 10^6) \div (2 \times 10^2)$

Order of Magnitude Approximations

In estimations, we can use scientific notation to help us solve 'order of magnitude' or 'Fermi' approximation problems. The main point of these approximations is to get your answer to be within an *order of magnitude*.

An order of magnitude refers to the value of the exponent that a number has when written in scientific notation. For example, 2,000,000 is a number on the order of magnitude of 6.

$$2,000,000 = 2 \times 10^6$$

An example of a correct order of magnitude approximation for the size of this class would be 70 students. There are actually only 35 students enrolled in this class, but because both 35 *and* 70 are within an order of magnitude (i.e. neither of the numbers are 10 times bigger or smaller than the other), this would be considered an accurate order of magnitude approximation.

You can think of these calculations as being the answer to the question 'How many digits will be in my final answer?'. There will always be one more digit than the order of magnitude.

Order of Magnitude	Exponent	Value
0	10^0	1 – 9
1	10^1	10 – 99
2	10^2	100 – 999
3	10^3	1,000 – 9,999

One classic example of an order of magnitude approximation is, 'If every human on Earth held hands and formed a chain wrapping around the equator, how many times would they circle the Earth?'

Do you think this will be on the order of magnitude of 0? 1? 2? 3? More? Using a few facts, and a couple rough estimates, we should be able to find a reasonably accurate answer!

(i) First lets look at what information we know:

- There are about 7,000,000,000 or 7×10^9 people who live on Earth.
- The radius of the Earth is about 6,000 or 6×10^3 kilometers.
- The average human arm span is about 1 meter.

All three of the numbers above are not very accurate, but we can ignore the nitty-gritty details when considering order of magnitude approximations.

(ii) Now we just have to do some calculations!

$$\begin{aligned} Length\,Of\,Humans &= (7 \times 10^9)\,people \times \frac{1\,meter}{1\,person} \\ &= 7 \times 10^9\,meters \end{aligned}$$

$$\begin{aligned} Circumference\,Of\,Earth &= 2 \times \pi \times (6 \times 10^3)\,kilometers \\ &\approx 2 \times 3 \times (6 \times 10^3)\,kilometers \\ &\approx 4 \times 10^4\,kilometers \end{aligned}$$

(iii) If we take a ratio of these two distances, we can see how many times bigger one is than the other.

$$\frac{Length\,Of\,Humans}{Circumference\,Of\,Earth} = \frac{7 \times 10^9\,meters}{4 \times 10^4\,kilometers} = \frac{7 \times 10^6\,kilometers}{4 \times 10^4\,kilometers} \approx 2 \times 10^2 = 200$$

This means humans could approximately wrap around the Earth 200 times. The actual answer is closer to 350 when using more precise values.

Example 4:

(a) Another example of an order of magnitude approximation would be 'How many Rubik's Cubes could fit inside this room?'

(i) Gather all of the information you can come up with.

- This is room is approximately the shape of a ____________________.
- This room approximately has a height of __________ meters, a width of __________ meters and a length of __________ meters.
- About __________ Rubik's Cubes could fit inside of a cubic meter.

(ii) From this information, I would estimate that ____________________ Rubik's Cubes can fit inside of this room.

(b) One final example of an order of magnitude approximation would be 'What volume of water is lost by a leaky faucet every year?'

(i) Gather all of the information you can come up with.

(ii) From this information, I would estimate that ____________________ liters of water are wasted by a leaky faucet each year.

Now that we have tried a few of these types of problems, I want everyone to try *estimating* how many of your handout can fit in the box at the front of the room.

I estimate that about ____________________ of this handout could fit into the box at the front of the classroom.

Wisdom Of The Crowd

An interesting phenomenon is statistics is known as the *wisdom of the crowd* effect. What this property tells us, is that a large group of individuals' answers, when averaged together, are generally found to be more accurate than any given individual's single answer.

As an experiment, lets see if the average of all of our *guesses* is more accurate than our estimations when we take into account the wisdom of the crowd!

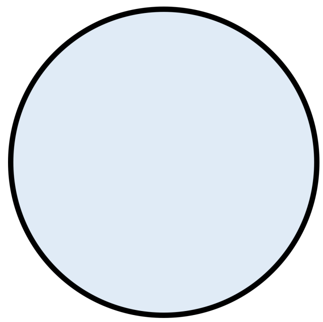
(*Use the space below to calculate the average of all of your classmates' guesses!*)

Number Of Handouts		
My Guess	My Estimation	Wisdom Of The Crowd

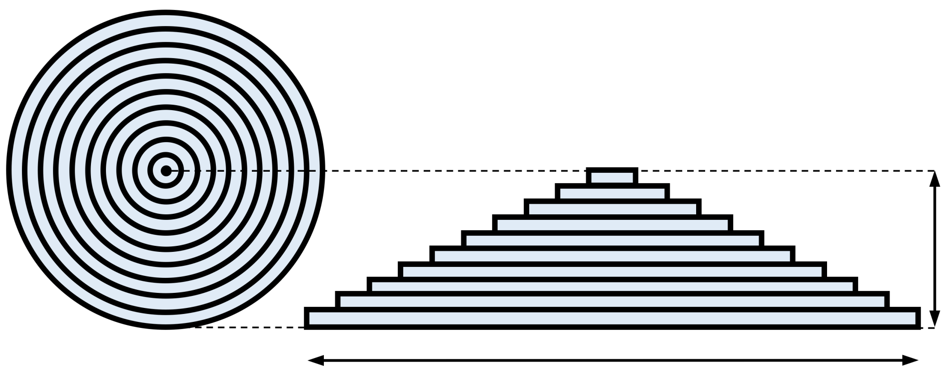
Estimating Areas

Now that we've covered estimating different kinds of arithmetic, we can finally try some more interesting things! In the next example, I will show you how you can derive the formula for the area of a circle ($A = \pi \times r^2$) using two different estimations!

Method 1: (i) Consider the circle below. Imagine taking this circle and dividing it into many small concentric rings.



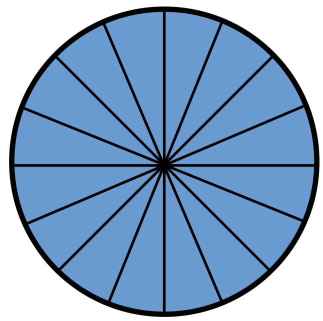
(ii) Picture taking each of these rings and unrolling them so they are perfectly straight. Each ring will approximately take the shape of a rectangle.



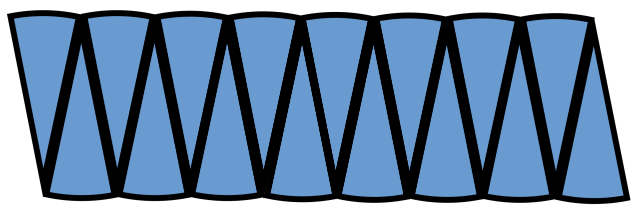
(iii) If you stack these rectangles on top of each other, they approximately form a triangle. The area of a triangle is $Area = \frac{1}{2} \times base \times height$. In this case, the height will be the radius of the circle, and the base will be the circumference of the circle. This gives us:

$$Area = \tfrac{1}{2} \times base \times height = \tfrac{1}{2} \times R \times 2\pi R$$
$$= \pi R^2$$

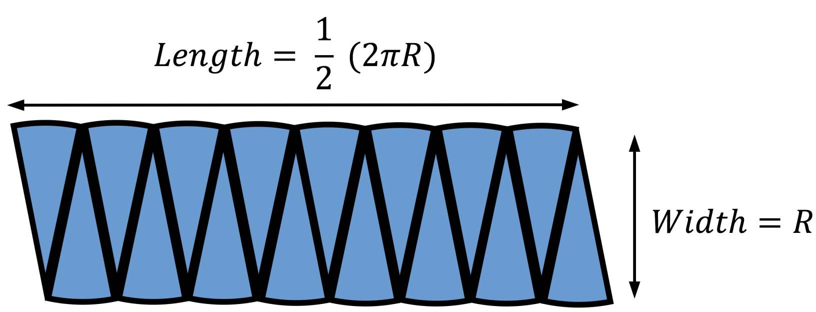
Method 2: (i) Imagine taking the same circle as in the previous method, except now instead of dividing it into rings, divide it into many small wedges.



(ii) Imagine taking these wedges, and arranging them to point vertically upwards and downwards in an alternating pattern as shown:



(iii) This approximately forms a rectangle. The width of the rectangle will be the radius of the circle. The length will be half the length of the circumference of the circle.

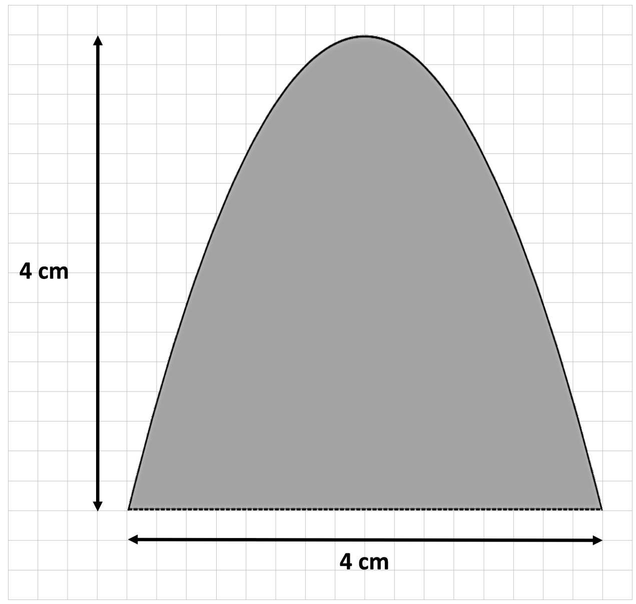


(iv) Calculating the area of this 'rectangle' gives us:

$$
\begin{aligned}
Area &= Length \times Width \\
&= \tfrac{1}{2} \times 2\pi R \times R \\
&= \pi R^2
\end{aligned}
$$

Example 5:

Using similar ideas to the ones used in the area of a circle approximations, estimate the area of the following shape:



Problems

1. How many digits would each of these numbers have?

(a) $(\sqrt{10})^{10}$

(b) 5^{10}

(c) 2^{500}

(d) π^{124}

(e) 7^{12}

2. What will be the order of magnitude for each of these calculations? You can use any facts, guesses and estimates that seem reasonable to you.

(a) If you drew one long straight line with a new pen, how long would it be when you run out of ink?

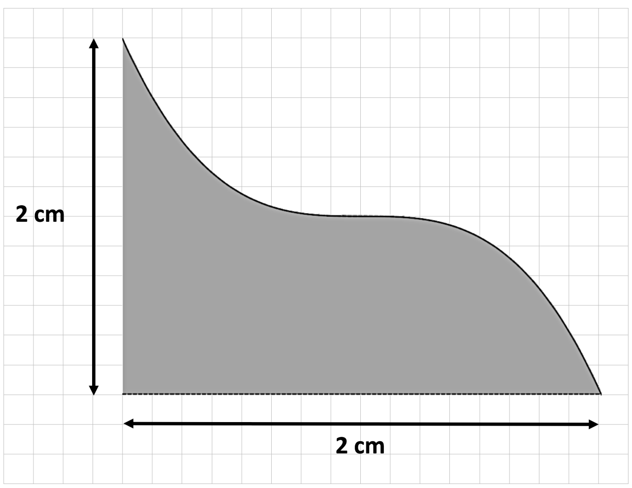
(b) How many times would you have to fold a sheet of paper in half for it to reach to the moon?

(c) How fast does human hair grow in centimeters per day?

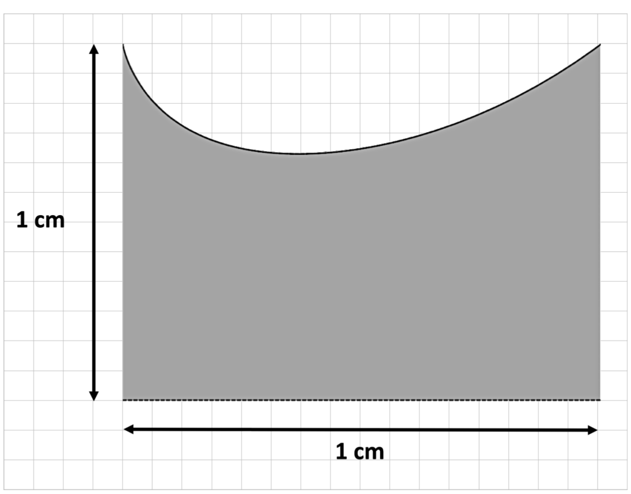
(d) What is the mass of the Earth?

(e) How many trees are there on the entire planet?

3. Estimate the areas of the following shapes:



(a)



(b)