Dynamic Supplier Contracts under Asymmetric Inventory Information

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Abstract

In this paper, we examine a supply chain in which a single supplier sells to a downstream retailer. We consider a multi-period model, with the following sequence of events. In period t the supplier offers a contract to the retailer, and the retailer makes her purchasing decision in anticipation of the demand. The demand then unravels and the retailer carries over any excess inventory to the next period (unmet demand is lost). In period $t+1$ the supplier designs a new contract based on his belief of the retailer's inventory, and the game is played dynamically. We assume that *short-term* contracts are used – i.e., the contracting is dynamic and is done at the beginning of each period. We also assume that the inventory position of the retailer before ordering is not observed by the supplier. This setting describes scenarios in which the downstream retailer does not share inventory/sales information with the supplier. For instance, it captures the phenomenon of retailers distorting past sales information to secure better contracting terms from their suppliers. In this setting, under certain assumptions, we characterize and evaluate the supplier's optimal contract. To do so, we cast our problem as an adverse-selection model with dynamic contracting. We show that, given relatively high production and holding costs, the optimal contract takes the form of a batch-order contract, which minimizes the retailer's information advantage. We then analyze the performance of the optimal contract with respect to some useful benchmarks and quantify the value of optimal contracting and the value of inventory information to the entire system. Dynamic adverse-selection models which are Markovian (that is, the action in a period affects the hidden state in the subsequent period) are recognized as being theoretically difficult and are thus relatively less understood. We believe that in our analysis we provide a framework for analyzing such models under short-term contracting and take an important first step towards understanding such models.

1 Introduction

Consider a two-echelon supply chain in which a retailer ("she") buys inventory from an upstream supplier ("he") in anticipation of random demand. The supplier decides on the type of contract and its terms, naturally, subject to the retailer's participation. Numerous studies have analyzed various important phenomena in this setting using the above model in which all information is public knowledge and there is exactly one period. Broadly speaking, this stream of research analyzes what are referred to as "selling to the newsvendor" models. Important issues that have been analyzed include supply chain coordination (Pasternack 1985, Cachon 2003, etc.), quantifying the loss to the system under commonly used contracts (i.e., the price of anarchy – Lariviere and Porteus 2001, Perakis and Roels 2006) and various other contracting issues.

In this paper, we make two assumptions that enrich this relatively well-understood model. First, we look at a standard multi-period inventory model and assume that *dynamic short-term* contracts are used by the players. Thus, in any period t, the supplier offers a purchasing contract to the downstream retailer, who may choose to buy in anticipation of demand. Once the purchasing decision is made by the retailer, inventory from the supplier is immediately transferred to the retailer and payments are received as per the contracting terms. Then, the demand in period t unravels. The retailer carries over excess inventory, if any, to the subsequent period $t+1$ (we assume unsatisfied demand is lost). In period $t+1$, the supplier offers a new contract to the retailer. Second, we assume that the sales at the retailer in any period are unobservable by the supplier. Since the supplier knows the distribution of the demand and the quantity purchased in period t, he can merely infer the distribution of the retailer's beginning inventory in period $t+1$. Thus, in any period, the supplier has imperfect information about the retailer's beginning inventory and factors this in when designing the contract. As a result, we analyze a dynamic adverse selection model in which the dynamics are Markovian, i.e., action taken by the retailer in period t affects the hidden information in period $t+1$.

We believe that these extensions to the single-period model are important and realistic. Our motivation for doing so is twofold. First, a multi-period model introduces dynamics in the analysis of contracting that is typically absent in a single-period analysis. Even in the simplest settings, several interesting phenomena have been documented. For instance, Anand et. al. (2006) consider a two-echelon supply chain in a multi-period setting similar to ours, but with two important distinctions. In their model, demand is price sensitive and deterministic and all information is public knowledge. They show, for instance, that the retailer carries inventory from one period to another and the entire supply chain benefits by him doing so. This result is dramatic as one may expect that, in the absence of nonlinear costs and uncertainty of any kind, inventory would be absent. Inventory in their model arises purely due to strategic considerations – by carrying inventory, the retailer is able to force the supplier to give him better wholesale prices. We expect that in our setting, due to information asymmetry and dynamic contracting, the resulting strategic interactions will yield further important insights. Second, our specific assumption on information asymmetry – i.e., that the supplier cannot observe the sales at the retailer in any period t and thus does not know the retailer's inventory position in period $t + 1$ – is very realistic. This setting encompasses a variety of situations in which retailers do not share their sales information with their suppliers and thus leave the suppliers in the dark about the exact purchasing requirements of the retailers. Indeed, numerous articles in the business press tout the potential value of retailers sharing their sales and inventory information with upstream players and lament the fact that, despite significant advancements in information technology, retailers are reluctant to do so. One comprehensive study (Statistics Canada 2003) of the Canadian logistics industry indicates that only about ten percent of Canadian retailers share inventory data over established web platforms. Reasons cited to explain this phenomena include a general lack of trust, confidentiality issues, and various strategic considerations by the retailers. We provide several references to this phenomena in the next section.

In this paper, we analyze the aforementioned model, in which contracting is dynamic and short-term and the action by the retailer in period t affects the state (unobserved by the supplier) in period $t + 1$. We thus study a dynamic contracting model using the principle-agent framework. A summary of our main results are as follows. First, in the single-period setting, the supplier's optimal contract resembles a quantity discount scheme. Further, the supplier prefers to deal with retailers whose initial inventory levels are small. Thus, one can say that the willingness of the supplier to trade with a retailer increases with the magnitude of the retailer's past sales. The magnitude of this willingness depends on the shape of the demand distribution. This phenomena is related to the "distortion from the bottom" effect observed in static adverse selection models (the "bottom" in our model corresponds to the highest inventory level). To better illustrate this phenomena, we explicitly calculate the optimal single-period contract with specific demand distributions.

Next, we extend the analysis to a multi-period setting in which we analyze the structure of the optimal contract. We first analyze a two-period model under exponential demand, which generates important insights on the structure of the optimal contracts. We then turn our attention to the infinite-horizon problem. Among the many reasons to analyze the infinite horizon, an important one is that insights gleaned from finite-horizon models may be tarnished by end-of-horizon effects. We analyze the infinite-horizon case assuming that the demand distribution is exponential. We chose the exponential form of demand for several reasons. An important one is that the exact analytical form of the optimal infinite-horizon contract is clean and its derivation is elegant in this case. Further, the extant operations management literature has used the exponential form of demand to get some traction on difficult problems when the analysis and insights are tricky to obtain. In particular, in dynamic models in which demand (Iglehart 1964) or sales (Lariviere and Porteus 1999) information is unknown and updated periodically, the exponential family of distributions is used. Some other examples include Cachon and Zhang (2006), Nagarajan and Rajagopalan (2007), Lau and Lau (1998) and Greenberg (1964). We show that when demand is exponential and the cost parameters are in a certain critical region of interest (we refer to this as the "high-cost region," to be made precise later), the optimal infinite-horizon contract is a "batch-order contract" (BOC). That is, the supplier gives a take-it-or-leave-it offer to the retailer wherein a fixed quantity b can be purchased for a payment of s. The only occasion in which a retailer accepts this contract is when her inventory position is zero, which happens infinitely often. The specific parameters b and s can be directly computed.

Given the mathematical difficulty of analyzing our problem, we take an approach similar in spirit to papers that study limiting regimes of difficult stochastic control problems. Limiting regimes yield insights on the structure of optimal policies. Policies constructed using this structural insight are empirically shown to perform well in non-limiting regimes. Similarly, for our problem, in scenarios in which either the costs do not fall in the above region or demand is not exponential, we empirically demonstrate that optimizing over the class of batch-order contracts (BOCs) does extremely well for the supplier. An interesting and useful property of the BOC is that the finite-horizon profit under the optimal BOC converges to the infinite-horizon profit function. This offers some measure of the stability of using the BOC. Observe that in an adverse selection setting such as ours, the stability of a contract form is important but not immediate from known results in dynamic programming. This is because of the complexity caused by the supplier's belief distribution which needs to be updated in every period. We note that even in static settings with information asymmetry in which players make inventory or capacity decisions, the structure of the optimal contract is often that of a nonlinear menu of contracts whose ease of implementation may raise some questions. Cachon and Zhang (2006), for instance, analyze simple contracts that perform extremely well when the optimal contract structure is complicated. An exception to this paradigm is a paper by Taylor and Xiao

(2006), in which the optimal contracts have elegant and insightful structures. Thus, the optimal characterization of a BOC in a dynamic setting such as ours may have some additional appeal.

We also make the following important observation. Our analysis of the infinite-horizon model in certain cost domains produces results that may be structurally similar to those in the single period setting. However, it is misleading to think that this is because of some inherent myopia in our analysis. Dynamic adverse selection models such as ours are not myopic. Myopic policies (or myopic equilibria in games) turn out to be a sufficient description of dynamic problems when a certain notion of "reachability" (also known as feasibility of optimal actions) can happen with probability one. This notion is irrelevant and absent from our analysis because the belief distribution is dynamically updated. We mention this here lest readers who see a similarity in structure (which happens only in certain cost domains) are tempted to think that a myopic solution sufficiently describes the dynamic game.

The rest of the paper is organized as follows. We first provide a brief literature review in §2, and in §3 we analyze the single-period model. In §4 we formalize the multi-period model and pay special attention to the two-period model with exponential demand. In §5, we derive the optimal contract for the infinite-horizon model with exponential demand and high costs, and numerically test it against other commonly used contracts when the demand distribution and cost parameters are more general. We conclude with a summary and discussion in §6. Finally, due to paucity of space, all proofs of relevant results, along with several useful pieces of analysis, are relegated to a fairly large online technical supplement. We have organized the supplement as follows. Appendix A contains all the proofs for the single-period model described in §3, along with optimal contracts for the exponential and uniform demand distributions; Appendix B contains all the proofs related to the multi-period models examined in §4 and §5; and Appendix C includes an in-depth analysis of a special multi-period model with only two periods and zero initial inventory.

2 Literature Review

There are a few streams of literature that are relevant to this paper. The first stream involves papers that provide evidence to the fact that retailers do not share inventory information with upstream suppliers and the various reasons for such actions. Strategic reasons against revealing truthful information manifest themselves in many ways. The well-known bullwhip effect (Lee et al. 1997) arises in part due to shortage gaming by retailers. Retailers may also choose to underreport past sales to elicit steep discounts from suppliers. Moreover, it is quite possible that retailers contemplate the possibility that if all sales and inventory information is shared, suppliers may use the knowledge of the retailers' purchasing requirements to manipulate wholesale prices or prioritize replenishment schedules based, for instance, on the relative importance of retailers. Lee and Whang (2000) describe several hurdles against information sharing. Other papers that discuss various aspects of the pitfalls of information sharing and reluctance of retailers to share information include Fawcett et al. (2006), Stank et al. (2002), Simatupang and Srdiharan (2006), Hart and Saunders (1993), Feldberg and van der Hejden (2003), and many others. An interesting stream of research pioneered by Desphande et al. (2006) recognizes the above fact (i.e., the reluctance of retailers to share purchasing requirements) and examines mechanisms that use the idea of *secure protocols*. These mechanisms allow for a free exchange of private information without actually disclosing it. Thus, these mechanisms may offer a way by which certain trust issues that besiege information sharing in supply chains may be resolved.

The second stream involves dynamic principal-agent games. This topic is of great interest to economists and its potential application to operations management is vast. However, the theory is still immature and thus the extant literature has seen few papers that tackle applications in operations management. Perhaps the first and only applications thus far are papers by Plambeck and Zenios (2000, 2003), in which dynamic moral-hazard problems are analyzed. Moral-hazard problems arise typically when the agent is risk averse (or with limited liability), unlike adverse-selection problems that often deals with risk-neutral agent. Dynamic adverse-selection problems are fraught with a host of well-known technical and expositional difficulties. Studies primarily focus on one of the following two paradigms – the hidden state is either constant (here the agent observes a realization exactly once, in period 1, unknown to the principal, and thereafter the state is unchanged) or the state is sampled from time-independent distributions (Salanie 1997, Bolton and Dewatripont 2005). These restrictive intertemporal information structures facilitate the analysis of the models. For instance, the optimal long-term contract either simply replicates the static contract (in the constant-information case) or gives a padded structure with an optimal static contract followed by static first-best contracts (in the setting with independently distributed random variables). These models do not account for a crucial and important phenomenon when the action taken by the agent in a status-quo period affects the state distribution in the subsequent period(s). Thus, intuition gained from these models may hold little value in more dynamic settings such as the one that we are interested in.

The model examined in this paper can be viewed as a special case of the dynamic adverse-selection model proposed by Zhang and Zenios (2007). While they study long-term contracts, we study short-term ones, which is a natural setting in many supply-chain contexts. The methodologies and results under the two contracting modes are drastically different. The main result of the short-term contracting literature is the "Ratchet effect" – because the principal can exploit the information revealed in an early period, the agent will be reluctant to reveal the true information early on, leading to a lot of pooling in the early period(s). To the best of our knowledge, this body of literature has mainly dealt with two-period models and the majority of them assume only two agent types. For instance, Freixas, Guesnerie, and Tirole (1985) study a problem between a central planner and a firm, where the firm has private information about its production efficiency (which can take two possible values) and the central planner offers a short-term contract in each of two periods to maximize social welfare. Laffont and Tirole (1988, 1990, 1993) also study a two-period short-term contracting problem between a regulator and a firm with private production costs. They show that the optimal contract is very complicated and involves a lot of pooling in the first period.

The operations literature has seen several papers that deal with inventory and capacity decisions in the presence of adverse selection. Illustrative examples are Corbett and de Groote (2000) and Corbett et al. (2004), in which suppliers are not privy to the cost structure of the buyer and optimal contracts for the supplier turn out to be quantity discount contracts, and Cachon and Zhang (2006), which study a queueing model with information asymmetry on costs. The above papers are static models; there are a few papers that look at supply chain contracts in multi-period settings. The paper by Anand et al. mentioned earlier is an example in which the dynamics are the closest to our work, with the crucial difference that their paper assumes complete public information. There is a stream of research that analyzes relational contracts (see, for instance, Taylor and Plambeck 2007a, 2007b) in which the emphasis is on incomplete contracts and repeated interactions. Thus, to the best of our belief, the operations literature has not seen a truly dynamic adverse selection model such as ours.

3 Single-Period Model

We start by looking at a single-period model. The upstream supplier sells a product to a retailer who faces random consumer demand. The supplier decides the contract and offers the terms of trade to the retailer, and the retailer decides the order quantity. The distribution of the demand is known to both parties, with cumulative distribution function (c.d.f.) $F(\cdot)$ and $\overline{F}(\cdot) = 1 - F(\cdot)$. The retailer privately owns an initial inventory, $x \geq 0$, which cannot be observed by the supplier, but the supplier knows its distribution, described by a c.d.f. $G(\cdot)$. All other information on demand forecasts, costs, revenues, etc. are public information. The retail price, r, is fixed, and the supplier's unit production cost is c. Thus, this is the well-known "selling to the newsvendor" model, with the extra assumption that the retailer's initial inventory is unknown to the supplier. Throughout this paper, we consider models with lost sales and zero lead times. This has important implications for the distribution of the initial inventory – in the presence of lost sales, the retailer's initial inventory distribution has a point mass at zero.

3.1 General Solution

We assume no salvage value at the end of the horizon. Given the *post-order inventory level* y_1, the retailer's single-period revenue function obtained through sales is given by $v_1(y_1) = rE[\min\{y_1, D_1\}] = \int_0^{y_1} r\overline{F}(\xi)d\xi$. Clearly, $v_1(y_1)$ is increasing and concave: $v_1'(y_1) = r\overline{F}(y_1) \geq 0$, and $v_1''(y_1) = -rf(y_1) \leq 0$. The property $v_1''(y_1) \leq 0$ implies $\frac{\partial^2 v_1(x_1+q_1)}{\partial x_1 \partial q_1} \leq 0$, which is the so-called *single-crossing property* in the literature. Suppose the initial inventory distribution $G(x_1)$ is defined over a bounded interval $[0, y_0]$ with $0 \leq y_0 \leq +\infty$. We assume the probability density function (p.d.f.) $g(x_1) > 0$ over $(0, y_0]$ and $G(0) \geq 0$ for generality. When the single-period problem is construed as the last period of a finite-horizon model with lost sales, the initial inventory x_1 is the result of the previous period's sales, i.e., $x_1 = (y_0 - D_0)^+ \equiv \max\{y_0 - D_0, 0\}$, where y_0 is the previous period's post-order inventory level. The distribution of x_1 contains a point mass at 0 in that case, i.e., $G(0) > 0$.

The sequence of events is as follows. First, the supplier proposes a menu contract $\{s_1(x_1), q_1(x_1)\}_{x_1 \in [0,y_0]}$ consisting of the *quantity plan* $q_1(x_1)$ and *payment plan* $s_1(x_1)$. If the retailer accepts the contract, she will report her initial inventory x_1 at the beginning of the period, which will trigger the order quantity $q_1(x_1)$ and payment $s_1(x_1)$ (Alternatively, the retailer directly selects a quantity-and-payment pair from the menu). Because the inventory cannot be observed by the supplier, he must provide incentives for the retailer to reveal the true x_1. The supplier's problem can be written as:

$$\max_{\{s_1(x_1), q_1(x_1)\}} \int_0^{y_0} \{s_1(x_1) - cq_1(x_1)\} dG(x_1) \tag{1a}$$

$$\text{s.t. } v_1(x_1 + q_1(x_1)) - s_1(x_1) \geq v_1(x_1 + q_1(\widehat{x}_1)) - s_1(\widehat{x}_1), \qquad x_1, \widehat{x}_1 \in [0, y_0] \tag{1b}$$

$$v_1(x_1 + q_1(x_1)) - s_1(x_1) \geq v_1(x_1), \qquad x_1 \in [0, y_0]. \tag{1c}$$

The constraints (1b) are the *incentive compatibility (IC)* constraints and (1c) are the *participation* (or *individual rationality, IR*) constraints. The IC constraints induce the retailer to report the

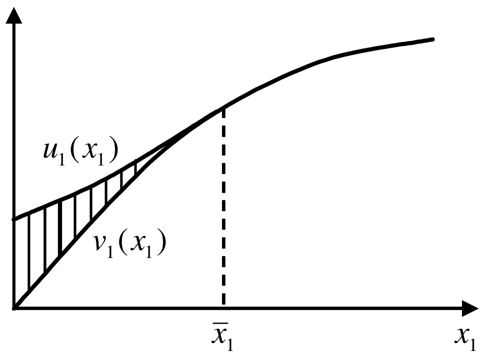

Figure 1: Retailer's profit as a function of x_1 with or without purchasing. The shaded area represents her information rent $u_1(x_1) - v_1(x_1)$.

true state x_1; the IR constraints ensure that choosing $(s_1(x_1), q_1(x_1))$ is at least as good as no transaction.

The retailer's net profit, as a function of the initial inventory x_1, is given by

$$u_1(x_1) \equiv v_1(x_1 + q_1(x_1)) - s_1(x_1). \tag{2}$$

The IC constraints (1b) are equivalent to $u_1(x_1) = \max_{\widehat{x}_1}\{v_1(x_1 + q_1(\widehat{x}_1)) - s_1(\widehat{x}_1)\}$. Assume $q_1(x_1)$ continuous over $(0, y_0]$ (which can be verified later). By the *envelope theorem*, we obtain the following local IC constraints:

$$u_1'(x_1) = v_1'(x_1 + q_1(x_1)),\ x_1 \in (0, y_0], \tag{3}$$

or,

$$u_1(x_1) = u_1(y_0) - \int_{x_1}^{y_0} v_1'(\widehat{x}_1 + q_1(\widehat{x}_1))d\widehat{x}_1,\ x_1 \in [0, y_0]. \tag{4}$$

The fact that $u_1(x_1)$ is continuous at $x_1 = 0$ (where $q_1(x_1)$ may be discontinuous) is implied by the IC constraints (1b), as shown in the proof of Theorem 1 in Appendix A.

A standard analysis of the above static problem involves noting that any incentive-compatible order plan $q_1(x_1)$ is (weakly) decreasing in x_1 and that the IR constraints are binding at y_0 and redundant at $x_1 \in [0, y_0)$ (see Lemma A1 in Appendix A). A special feature of this model is that the reservation profit function $v_1(x_1)$ in the IR constraints is increasing (and concave) in x_1, which makes the retailer with a higher inventory a "worse type" because an additional unit has less value to the retailer when x_1 is larger. Thus, the *information rent* $u_1(x_1) - v_1(x_1)$ (the extra profit yielded to the retailer in exchange for her private information) decreases in x_1, as illustrated in Figure 1.

After replacing $s_1(x_1)$ by $v_1(x_1 + q_1(x_1)) - u_1(x_1)$ and using equation (4), we can rewrite the objective function (1a) as

$$\int_{0^+}^{y_0} J_1(x_1, q_1(x_1))g(x_1)dx_1 + J_1(0, q_1(0))G(0) - u_1(y_0),$$

where $\int_{0^+}^{y_0} h(x_1)dx_1$ is the short-hand notation for $\lim_{\underline{x}_1 \to 0^+} \int_{\underline{x}_1}^{y_0} h(x_1)dx_1$ and $J_1(x_1, q_1)$ is defined as

$$J_1(x_1, q_1) \equiv \begin{cases} v_1(q_1) - cq_1, & x_1 = 0, \\ v_1(x_1 + q_1) - cq_1 + v_1'(x_1 + q_1)\frac{G(x_1)}{g(x_1)}, & x_1 \in (0, y_0]. \end{cases} \tag{5}$$

The function $J_1(x_1, q_1)$ isolates the part of the supplier's profit that is directly affected by the order quantity $q_1(x_1)$, and is called *virtual surplus* in the literature. The effect of $q_1(x_1)$ is two-fold, consisting of an internal effect and an external effect. First, it affects the channel profit when the initial inventory is x_1, which is given by $(v_1(x_1 + q_1(x_1)) - cq_1(x_1))g(x_1)$. Second, it affects the retailer's profit, and hence the information rent, when the initial inventory is lower than x_1 (the retailer's profit at x_1 is unaffected because of the payment $s_1(x_1)$). By the local IC constraints (4), the term $v_1'(x_1 + q_1(x_1))$ pulls the retailer's profit downward over the entire range $[0, x_1)$, resulting in a total gain of $v_1'(x_1 + q_1(x_1))G(x_1)$ for the supplier. Thus, the expression of $J_1(x_1, q_1)$ for $x_1 \in (0, y_0]$ follows. On the other hand, the order quantity $q_1(0)$ has no impact on the information rent for any initial inventory. Therefore, apart from the standard models in the literature, our model requires different forms of virtual surplus for $x_1 = 0$ and $x_1 > 0$, which is the result of the point mass at zero inventory and is an important feature of the model. Consequently, as will soon be seen, the quantity plan $q_1(x_1)$ may be discontinuous at 0.

The optimal quantity plan, $q_1(x_1)$, can then be determined from the first-order condition (FOC) $\partial J_1(x_1, q_1)/\partial q_1 = 0$, given x_1. We state the main results of the single-period model in the following two theorems and leave the proofs and other technical details to Appendix A. Our first result identifies a set of sufficient conditions for the first-order solution to be optimal. Before stating the result, we note that if the retailer's initial inventory can be observed by the supplier, the optimal (first-best, FB) order-up-to level is $y_1^* = F^{-1}(\frac{r-c}{r})$.

Theorem 1 *Under conditions*

$$\frac{g(x_1)}{G(x_1)} + \frac{f'(x_1 + q_1)}{f(x_1 + q_1)} \geq 0, \text{ for } x_1 > 0 \text{ and } q_1 > 0, \tag{6}$$

$$\text{and } \frac{d}{dx_1}\left(\frac{G(x_1)}{g(x_1)}\right) \geq 0, \text{ for } x_1 > 0, \tag{7}$$

the supplier's optimal order plan $q_1(x_1)$ *satisfies: (1)* $q_1(0) = F^{-1}(\frac{r-c}{r})$*; (2) for* $x_1 \in (0, y_0]$, $q_1(x_1)$ *solves the first-order condition*

$$F(x_1 + q_1) + f(x_1 + q_1)\frac{G(x_1)}{g(x_1)} = \frac{r-c}{r} \tag{8}$$

if it has a positive solution, otherwise $q_1(x_1) = 0$*. Under this plan,*

$$\lim_{x_1 \to 0^+} q_1(x_1) \begin{cases} = q_1(0), & \text{if } G(0) = 0; \\ < q_1(0), & \text{if } G(0) > 0. \end{cases}$$

The conditions in the theorem are worth further discussion. Condition (6) is basically the second-order condition that ensures $J_1(x_1, q_1)$ to be maximized by $q_1(x_1)$ at x_1. It is a common condition in the adverse-selection literature, although it may assume different forms in various models. This condition involves $v_1'''(y_1)$, the third derivative of the retailer's revenue function, because the virtual surplus $J_1(x_1, q_1)$ already involves its first derivative, $v_1'(y_1)$. Condition (7), along with (6), ensures that $q_1(x_1)$ is weakly decreasing in x_1. It is similar to the standard "increasing hazard rate" condition in the literature, but in the opposite direction, because the types in our model (values of x_1) are ordered in an opposite way from those in a standard model. These two conditions constitute a sufficient, but not necessary, set of conditions, and are satisfied by several common distributions. However, to be prudent, these conditions should be verified when the solution is found. All models and examples discussed in this paper and the appendices have passed this test. For instance, in the exponential demand case with p.d.f. $f(\xi) = \lambda e^{-\lambda\xi}$, $\xi \geq 0$, when the initial inventory is derived from $x_1 = (y_0 - D_0)^+$, we have $G(x_1) = e^{-\lambda(y_0 - x_1)}$, $0 \leq x_1 \leq y_0$, and $g(x_1) = \lambda e^{-\lambda(y_0 - x_1)}$, $0 < x_1 \leq y_0$, and the two conditions reduce to $\lambda - \lambda \geq 0$ and $\frac{d}{dx_1}\left(\lambda^{-1}\right) \geq 0$, respectively.

Figure 2 illustrates the optimal quantity plans when $G(\cdot)$ is uniform over $[0, 1]$ (with no point mass) and $F(\cdot)$ is uniform, exponential or normal ($F(\cdot)$ is normalized so that $F^{-1}(\frac{r-c}{r}) = 5$). The order plan can be efficiently computed for most realistic scenarios. More details can be found in Appendix A.

We note the special structure associated with the state in which the initial inventory is zero. When the retailer reports this state, the first-best quantity is transacted. This agrees with the "efficiency at the top" phenomenon observed in static adverse selection problems ("top" corresponds to zero inventory in our setting). The inefficiency in the channel arises only due to the possibility of the retailer inflating her inventory levels to downplay the value of additional units. The supplier's optimal contract protects him from the retailer's actions and exhibits the standard feature of "downward distortion at the bottom." As discussed above, the order quantity $q_1(x_1)$ at any $x_1 > 0$ has an external effect on the retailer's profit at all $x_1' < x_1$. The term $v_1'(x_1+q_1)\frac{G(x_1)}{g(x_1)}$ in the expression of $J_1(x_1,q_1)$ draws $q_1(x_1)$ downward from the channel-optimal quantity, because $v_1'(x_1+q_1)$ is decreasing in q_1 ($v_1''(\cdot) \leq 0$). A side effect of this distortion is that, when $G(\cdot)$ contains a point mass at 0, $q_1(x_1)$ is discontinuous at 0 with a downward jump; if $G(0) = 0$, however, the term $v_1'(x_1+q_1)\frac{G(x_1)}{g(x_1)}$ vanishes at $x_1 = 0$ and $q_1(x_1)$ is still continuous. Later, in the multi-period setting, we will push this discontinuity to the extreme to create an extremely simple contract such that $q_t(x_t) = 0$ for all $x_t > 0$, which is a somewhat drastic take-it-or-leave-it offer by the supplier.

Next, we show some properties of the optimal order plan and payment plan.

Theorem 2 *Under the conditions of Theorem 1, the optimal contract exhibits the following properties:*

1. (Order plan) $q_1(x_1)$ *is weakly decreasing in* x_1*, and if (6) is satisfied strictly at* $x_1 > 0$*, then* $q_1'(x_1) \leq -1$ *on* $(0, \overline{x}_1]$ *for some* $\overline{x}_1 < y_1^*$ *such that* $q_1(\overline{x}_1) = 0$*;*
2. (Payment plan) *The optimal payment plan observes quantity discounting; that is,* $s_1(x_1)$ *is increasing and concave in terms of* $q_1(x_1)$*.*

Theorem 2 and Figure 2 show that the qualitative pattern of $q_1(x_1)$ is consistent across demand distributions and cost regions: $q_1(x_1)$ starts from y_1^*, decreases faster than $-x_1$, and hits zero at some $\overline{x}_1 < y_1^*$. The first part of the theorem also implies a threshold structure – the trade takes place if and only if the retailer reports an initial inventory less than $\overline{x}_1$.

In the next subsection, we derive the optimal contract when the demand is exponentially distributed and the single period is the last period of a finite-horizon problem. Thus, the initial inventory distribution is derived and not assumed. In Appendix A, we compute the optimal contracts in some other situations, including exponentially distributed demand with uniformly distributed initial inventory, and uniformly distributed demand with derived or uniformly distributed initial inventory.

3.2 Special Case: Exponential Demand

In this subsection, we assume the demand distribution has c.d.f. $F(\xi) = 1 - e^{-\lambda\xi}$ and p.d.f. $f(\xi) = \lambda e^{-\lambda\xi}$, $\xi \geq 0$. If we consider a lost-sales system and assume there is no salvage value for unsold items, a straightforward calculation reveals that the first-order derivative $\partial J_1(x_1,q_1)/\partial q_1$, for $x_1 > 0$, becomes

$$\frac{\partial J_1(x_1,q_1)}{\partial q_1} = r\left[1 - \lambda\frac{G(x_1)}{g(x_1)}\right]e^{-\lambda(x_1+q_1)} - c. \tag{9}$$

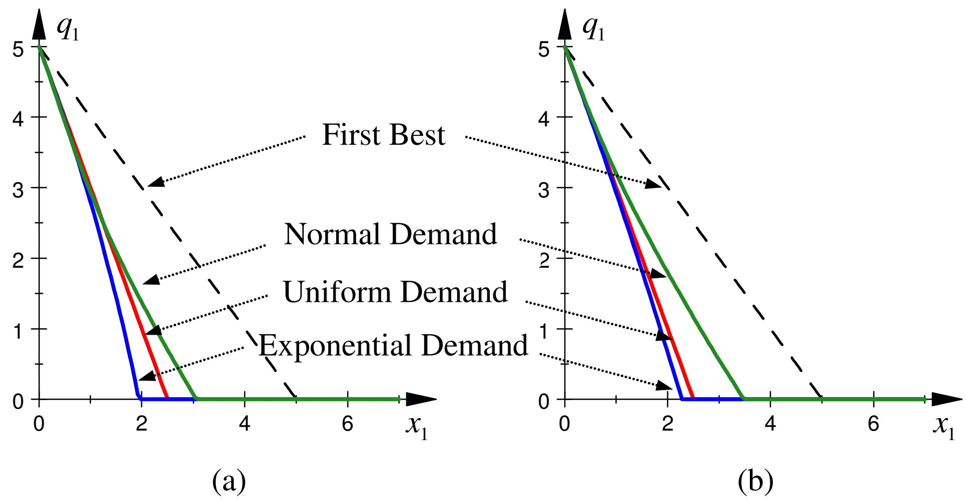

Figure 2: Optimal quantity plan $q_1(x_1)$ under uniform inventory distribution and uniform, exponential, or normal demand distribution in (a) the low cost case, $c = 0.2r$, and (b) the medium cost case, $c = 0.5r$.

Suppose the initial inventory is given by $x_1 = (y_0 - D_0)^+$, where D_0 also follows the distribution $F(\xi)$. Then, $G(x_1) = e^{-\lambda(y_0 - x_1)}$ for $0 \leq x_1 \leq y_0$, with a point mass at 0, and $g(x_1) = \lambda e^{-\lambda(y_0 - x_1)}$ for $0 < x_1 \leq y_0$. As a result, the first-order derivative (9) becomes $\partial J_1(x_1, q_1)/\partial q_1 = -c < 0$, and the optimal order plan and payment plan are

$$q_1(x_1) = \begin{cases} \lambda^{-1} \ln(\frac{r}{c}), & x_1 = 0 \\ 0, & x_1 \in (0, y_0] \end{cases} \quad ; \quad s_1(x_1) = \begin{cases} \lambda^{-1}(r - c), & x_1 = 0 \\ 0, & x_1 \in (0, y_0] \end{cases} .$$

Thus, the static optimal contract with exponential demand distribution and derived initial-inventory distribution exhibits a rather severe distortion due to information asymmetry. In such a case, the supplier will not deal with any retailer who reports a positive initial inventory, i.e., the threshold is $\overline{x}_1 = 0$. This is not always true, as can be seen from other examples in Appendix A.

4 Dynamic Model

We now analyze dynamic contracts in the multi-period problem with Markovian dynamics. In this section, we formulate the problem, discuss some basic facts about the model, and derive the optimal contract in the two-period case under exponential demand. This section prepares us for studying the infinite-horizon model in §5, which will be the emphasis of the paper. The structure of the optimal contract is more prominent in the infinite-horizon case – end-of-the-horizon effects sort themselves out and, at least for a special case, we are able to characterize the optimal contract.

The multi-period setting encompasses at least two well-known contracting modes, i.e., long-term and short-term contracting. Under long-term contracting, the principal can make a credible commitment to a contingency plan that covers the entire horizon. Under short-term contracting, the principal only offers a single-period contract in each period. We consider short-term contracts in this paper. Such a setting is appropriate if the principal prefers the relative simplicity of one-period contracts, or if he (she) lacks the credibility of carrying out a multi-period contract. In practice, both contracting schemes are used. A significant number of settings involve suppliers who do not provide retailers with detailed long-term price schedules. Moreover, given some of the technical and framing difficulties, a thorough understanding of short-term dynamic contracts with adverse selection seems to be a rather challenging task of significant theoretical interest.

An important issue related to short-term contracting is the definition of equilibrium. Conceivably, when designing the contract in period t given his belief about the retailer's beginning inventory x_t, the supplier must foresee the retailer's response in that period, which depends on the retailer's expectation of the contract to be offered in the next period that in turn depends on the supplier's belief about the beginning inventory in the next period, and so on. This leads to a complex mutual belief process. A proper equilibrium concept for our setting is the *perfect Bayesian equilibrium* (PBE), see e.g., Fudenberg and Tirole (1991) and Bolton and Dewatripont (2005). A suitable solution concept for short-term contracting needs the following required characteristics: (1) the contract offered by the supplier in period t maximizes his expected total profit-to-go given his belief about the beginning inventory x_t; (2) the retailer's response in period t maximizes her expected total profit-to-go given the contract in period t and those to be offered in all future periods; (3) the supplier's belief about the beginning inventory x_{t+1} is updated according to the Bayes' rule from his belief about x_t and the retailer's response in period t.

At the beginning of period t, the supplier designs the contract $\{s_t(x_t), q_t(x_t)\}$ to maximize his expected profit-to-go, with respect to the belief G_t of the beginning inventory x_t and subject to the retailer's IC and IR constraints (to be defined shortly). Following a typical way of finding a PBE, we start with an assumption of the structure of the contract in period t. In view of the optimal static contract, we postulate that the optimal contract in any period of a multi-period model is made up of two continuous regions: a *separating region*, denoted by $[0, \overline{x}_t]$, and a *pooling region*, denoted by $(\overline{x}_t, +\infty)$ ($\overline{x}_t$ depends on the belief G_t in general). The supplier will induce the retailer to reveal the true inventory x_t if x_t falls in the separating region, but will otherwise leave the retailer alone. In the pooling region, the order quantity is $q_t(x_t) = 0$. In the following analysis, we will focus on the separating region, because the pooling region is simply the complement of the former and the contract therein is trivially found.

4.1 Supplier's Problem in the Separating Region

First, we introduce some notation. Assume the demand D_t in every period follows the same distribution $F(\cdot)$ (i.e. i.i.d.). If the retailer orders q_t in period t, the next period's beginning inventory is given by $x_{t+1} = \max\{x_t + q_t - D_t, 0\}$, where $x_t + q_t$ is the post-order inventory level in period t. The retailer's expected revenue in period t from selling these $x_t + q_t$ units, adjusted by the cost of holding the leftovers to the next period, is captured by the function $v_t(x_t + q_t)$ (for the last period T, v_t need not include any inventory cost). Let $U_{t+1}(y_t \mid \widehat{y}_t)$ be the retailer's expected profit-to-go from period $t+1$ onward if the true post-order inventory level in period t is y_t yet the supplier's perception is $\widehat{y}_t$ (which determines the supplier's belief about x_{t+1} and thus the contract offered in period $t+1$). Let $\Pi_{t+1}(y_t)$ and $\Psi_{t+1}(y_t) = \Pi_{t+1}(y_t) + U_{t+1}(y_t \mid y_t)$ be the supplier and the channel's expected profits-to-go, respectively, from period $t+1$ onward given the true y_t. Future profits are discounted by a factor $\delta \in (0, 1)$ per period.

Then, the supplier's problem given belief G_t and over the region $[0, \overline{x}_t]$ is the following:

$$\max_{\{s_t(x_t), q_t(x_t)\}_{x_t \in [0, \overline{x}_t]}} \int_0^{\overline{x}_t} \{s_t(x_t) - cq_t(x_t) + \delta \Pi_{t+1}(x_t + q_t(x_t))\}\, dG_t \tag{10a}$$

$$\text{s.t. } v_t(x_t + q_t(x_t)) - s_t(x_t) + \delta U_{t+1}(x_t + q_t(x_t) \mid x_t + q_t(x_t))$$
$$\geq v_t(x_t + q_t(\widehat{x}_t)) - s_t(\widehat{x}_t) + \delta U_{t+1}(x_t + q_t(\widehat{x}_t) \mid \widehat{x}_t + q_t(\widehat{x}_t)), \; x_t, \widehat{x}_t \in [0, \overline{x}_t] \tag{10b}$$

$$v_t(x_t + q_t(x_t)) - s_t(x_t) + \delta U_{t+1}(x_t + q_t(x_t) \mid x_t + q_t(x_t))$$
$$\geq v_t(x_t) + \delta \underline{U}_{t+1}(x_t), \quad x_t \in [0, \overline{x}_t]. \tag{10c}$$

Note that the IC constraints (10b) only allow $\widehat{x}_t$ to vary in the separating region. To ensure full incentive compatibility, $\widehat{x}_t$ should be allowed in the pooling region $(\overline{x}_t, +\infty)$ as well. That can be easily done after the problem (10) is solved and thus is omitted from the formulation. The function $U_{t+1}(x_t + q_t(\widehat{x}_t) \mid \widehat{x}_t + q_t(\widehat{x}_t))$ in (10b) deserves a closer inspection: its second argument, $\widehat{x}_t + q_t(\widehat{x}_t)$, is the supplier's perception about the post-order inventory level in period t resulting from the reported beginning inventory $\widehat{x}_t$ and the observed order quantity $q_t(\widehat{x}_t)$; while its first argument, $x_t + q_t(\widehat{x}_t)$, is the actual post-order inventory. The function $\underline{U}_{t+1}(x_t)$ in the IR constraints (10c) gives the retailer's expected profit-to-go from period $t+1$ onward in the default setting, that is, ordering nothing from period $t+1$ onward.

Let $u_t(x_t)$ denote the retailer's expected profit-to-go from period t onward given beginning inventory x_t. Then we have

$$u_t(x_t) = v_t(x_t + q_t(x_t)) - s_t(x_t) + \delta U_{t+1}(x_t + q_t(x_t) \mid x_t + q_t(x_t)). \tag{11}$$

The IC constraints (10b) are equivalent to

$$u_t(x_t) = \max_{\widehat{x}_t \in [0,\overline{x}_t]} \{v_t(x_t + q_t(\widehat{x}_t)) - s_t(\widehat{x}_t) + \delta U_{t+1}(x_t + q_t(\widehat{x}_t) \mid \widehat{x}_t + q_t(\widehat{x}_t))\}. \tag{12}$$

Assume $U_{t+1}(y_t \mid \widehat{y}_t)$ continuous and differentiable (which can be verified after the solution is found) and define $y_t(x_t) = x_t + q_t(x_t)$. The envelope theorem implies

$$u_t'(x_t) = v_t'(y_t(x_t)) + \delta \frac{\partial U_{t+1}(y_t(x_t) \mid y_t(x_t))}{\partial y_t}, \tag{13}$$

where $\frac{\partial U_{t+1}(y_t(x_t) \mid y_t(x_t))}{\partial y_t}$ is an alternative expression of $\left.\frac{\partial U_{t+1}(y_t \mid \widehat{y}_t)}{\partial y_t}\right|_{y_t = \widehat{y}_t = y_t(x_t)}$. Then, the retailer's profit-to-go $u_t(x_t)$ can be pinned down to $u_t(\overline{x}_t)$ as follows:

$$u_t(x_t) = u_t(\overline{x}_t) - \int_{x_t}^{\overline{x}_t} u_t'(z)dz, \;\; x_t \in [0, \overline{x}_t]. \tag{14}$$

Similar to the static case, we define the following *virtual surplus* by taking future profits into account: for $x_t \in [0, \overline{x}_t]$,

$$J_t(y_t \mid x_t) \equiv \begin{cases} v_t(y_t) - cy_t + \delta\Psi_{t+1}(y_t), & x_t = 0, \\ v_t(y_t) - cy_t + \delta\Psi_{t+1}(y_t) + \left(v_t'(y_t) + \delta \frac{\partial U_{t+1}(y_t \mid y_t)}{\partial y_t}\right) \frac{G_t(x_t)}{g_t(x_t)}, & x_t \in (0, \overline{x}_t]. \end{cases}$$

The virtual surplus at x_t is basically the slice of the supplier's expected profit-to-go that is directly affected by the inventory target (y_t) chosen for x_t. The function takes different forms at $x_t = 0$ and $x_t > 0$, for the same reason as in the static case. We can show the following first-order result (see Appendix B), which will be crucial in our latter analysis in the high cost domain.

Proposition 1 *The optimal post-order inventory plan $y_t(x_t)$ must satisfy*

$$y_t = x_t \textit{ and } \frac{\partial J_t(y_t \mid x_t)}{\partial y_t} \le 0, \quad \textit{or} \quad y_t > x_t \textit{ and } \frac{\partial J_t(y_t \mid x_t)}{\partial y_t} = 0.$$

The functions $\Psi_{t+1}'(y_t)$ and $\partial U_{t+1}(y_t \mid \widehat{y}_t)/\partial y_t$ are needed for computing $\partial J_t(y_t \mid x_t)/\partial y_t$. They depend upon the demand distribution and can be computed on a case-by-case basis. Note that $U_{t+1}(y_t \mid \widehat{y}_t)$ is in fact recursively defined through expression (11) – it is just the expectation of $u_{t+1}(x_{t+1})$, in which the true distribution of x_{t+1} is determined from y_t while the underlying contract is based on $\widehat{y}_t$.

We emphasize that the supplier's belief G_t underlies everything discussed in this subsection – the optimal contract $\{s_t(x_t), q_t(x_t)\}$, the retailer's profit-to-go function $u_t(x_t)$, and the virtual surplus $J_t(y_t \mid x_t)$. This dependence is suppressed in the notation for the sake of simplicity.

4.2 Belief Process under Exponential Demand

The complex nature of the belief process is a main source of difficulty for the short-term contracting problem. In this subsection we show an important property of the belief process under exponential demand that will significantly simplify our subsequent analysis.

We assume that the demand in every period follows the c.d.f. $F(\xi) = 1 - e^{-\lambda\xi}$ and p.d.f. $f(\xi) = \lambda e^{-\lambda\xi}$, $\xi \geq 0$. If the post-order inventory in the previous period is y_{t-1}, the beginning inventory of the current period should be given by $x_t = (y_{t-1} - D_{t-1})^+$, which has c.d.f. $G_t(x_t|y_{t-1}) = e^{-\lambda(y_{t-1}-x_t)}$, for $0 \leq x_t \leq y_{t-1}$, and p.d.f. $g_t(x_t|y_{t-1}) = \lambda e^{-\lambda(y_{t-1}-x_t)}$, for $0 < x_t \leq y_{t-1}$. In that case, $\frac{G_t(x_t|y_{t-1})}{g_t(x_t|y_{t-1})} = \lambda^{-1}$ for $0 < x_t \leq y_{t-1}$. This property of the beginning-inventory distribution is generalized by the following definition:

Definition 1 *A distribution (c.d.f.) $G_t(x_t)$ defined on $[0, \overline{x}_t]$ is* ***weakly reverse exponential*** *(WRE) with rate λ if $\frac{G_t(x_t)}{g_t(x_t)} \geq \lambda^{-1}$ for $x_t \in (0, \overline{x}_t]$.*

We show that under exponential demand, regardless of the contract executed in period $t-1$, the distribution of the beginning inventory of period t is weakly reverse exponential. It is important to note that in order to derive the optimal contract through backward induction we need to allow arbitrary contract structures in the past. Thus, if the beginning inventory of the previous period follows an arbitrary distribution $G_{t-1}(x_{t-1})$ and an arbitrary contract was offered in that period, then the retailer's order quantity q_{t-1} may reveal an arbitrary set of beginning inventories in that period, denoted by S_{t-1}, due to the possibility of pooling. For instance, if the supplier offers in period $t-1$ a contract consisting of a separating region $[0, \overline{x}_{t-1}]$ and a pooling region $(\overline{x}_{t-1}, +\infty)$, the choice of $q_{t-1} = 0$ by the retailer only suggests that the beginning inventory x_{t-1} belongs to the pooling region $(\overline{x}_{t-1}, +\infty)$, i.e., $S_{t-1} = (\overline{x}_{t-1}, +\infty)$. In the separating case, S_{t-1} is a singleton set. Let $q_{t-1} + S_{t-1}$ denote the set of possible post-order inventories $\{q_{t-1} + x_{t-1} : x_{t-1} \in S_{t-1}\}$. We have the following result:

Theorem 3 *Suppose an arbitrary contract is offered in period $t-1$, and the retailer orders an arbitrary quantity $q_{t-1} \geq 0$ that implies the beginning inventory $x_{t-1} \in S_{t-1}$ for some set S_{t-1}. If the demand in period $t-1$ is exponential, given any belief $G_{t-1}(x_{t-1})$ of the beginning inventory of period $t-1$, the supplier's belief of the beginning inventory of period t, denoted by $G_t(x_t|y_{t-1} \in q_{t-1}+S_{t-1})$, is weakly reverse exponential.*

The theorem unveils a remarkable fact that the posterior belief $G_t(x_t)$ is WRE following any prior belief $G_{t-1}(x_{t-1})$ under exponential demand, which enables the optimal short-term contracts being constructed through backward induction when costs are high, as will soon be seen.

4.3 Batch-Order Contracts

As shown in the static case, if the belief $G_1(x_1)$ has a point mass at zero, the optimal order plan $q_1(x_1)$ is discontinuous at $x_1 = 0$ with a downward jump. An extreme example of this structure is the case in which the supplier does not sell to the retailer when x_1 is positive. In fact, this happens when (holding and production) costs are high or when the initial inventory x_1 is derived from $(y_0 - D_0)^+$ and D_0 is exponential. Here, the contract only offers two options, a fixed quantity or zero. We call such a simple contract a *batch-order contract* (BOC), formally defined by a quantity and payment pair (b_t, s_t) in period t: the retailer can obtain b_t units at the total price of s_t (if she reports zero inventory) or nothing (otherwise). More precisely, the contract $\{s_t(x_t), q_t(x_t)\}$ is of the following form:

$$q_t(x_t) = \begin{cases} b_t, & \text{if } x_t = 0 \\ 0, & \text{if } x_t > 0 \end{cases}, \quad s_t(x_t) = \begin{cases} s_t, & \text{if } x_t = 0 \\ 0, & \text{if } x_t > 0 \end{cases}.$$

If the contract is incentive compatible, the retailer should voluntarily place the order when her inventory hits zero. The BOC has some advantages – it is relatively easy to characterize, and easy to implement in practice (it has already been used in some industries, although its implementation may be driven by different reasons).

In the next subsection, we will extend the static result of the exponential-demand case to the last two periods of a finite-horizon model. We will show that the optimal contract in the last period is always a BOC, and that the same is true in the second-last period if the costs c and h are relatively high. A similar result can be shown for the third-last period and so on, but to avoid obscuring the focus of the paper, we will immediately move on to the infinite-horizon case in §5 and show that a stationary BOC (where $b_t \equiv b$ and $s_t \equiv s$ for all t) is optimal given relatively high costs.

4.4 Optimal Contracts in the Last Two Periods under Exponential Demand

In this subsection, we assume the demand is exponentially distributed with rate λ in every period. The cost of holding one unit of inventory for one period is denoted by h. In the T-period case, the retailer's expected revenue (minus inventory cost) in period t is given by

$$v_t(y_t) = \begin{cases} rE\min\{y_t, D_t\} = \lambda^{-1}r(1 - e^{-\lambda y_t}), & t = T, \\ rE\min\{y_t, D_t\} - hE(y_t - D_t)^+ = \lambda^{-1}(r+h)(1 - e^{-\lambda y_t}) - hy_t, & t \le T-1. \end{cases}$$

The special form of the revenue function in the last period is due to the absence of holding cost at the end of the horizon. Accordingly, the derivatives of $v_t(y_t)$ are given by:

$$v_t'(y_t) = \begin{cases} re^{-\lambda y_t}, & t = T, \\ (r+h)e^{-\lambda y_t} - h, & t \leq T-1, \end{cases} \quad v_t''(y_t) = \begin{cases} -\lambda r e^{-\lambda y_t}, & t = T, \\ -\lambda(r+h)e^{-\lambda y_t}, & t \leq T-1. \end{cases}$$

It can be shown that the optimal contract in the last period is a simple BOC.

Proposition 2 *Under exponential demand, the optimal contract in the last period of a finite-horizon problem is given by*

$$q_T(x_T) = \begin{cases} y_T^* = \lambda^{-1}\ln(\frac{r}{c}), & x_T = 0 \\ 0, & x_T > 0 \end{cases} ; \quad s_T(x_T) = \begin{cases} \lambda^{-1}(r-c), & x_T = 0 \\ 0, & x_T > 0 \end{cases} .$$

This result follows from Theorem 3 and implies that the weakly-reverse-exponential property of the belief G_T is enough to induce the BOC in the last period. Although this optimal contract is identical to the static contract derived in §3.2, it is non-trivial in the sense that it is independent of the history and serves as the basic step of the backward induction procedure in finding the sequence of optimal short-term contracts.

To derive the optimal contract in the second-last period, we first compute the expected channel profit and retailer's profit in the last period, $\Psi_T(y_{T-1})$ and $U_T(y_{T-1})$, given the true post-order inventory y_{T-1} in the second-last period. Because the optimal contract in the last period is independent of the supplier's belief G_T, the perceived post-order inventory $\widehat{y}_{T-1}$ in the second-last period has no role to play, which is why we only need $U_T(y_{T-1})$ instead of $U_T(y_{T-1} \mid \widehat{y}_{T-1})$. Without loss of generality, the terminal channel- and retailer-profit functions, $\Psi_{T+1}(\cdot)$ and $U_{T+1}(\cdot)$, are normalized to zero.

Lemma 1 *The expected channel profit and retailer's profit in the last period, given the post-order inventory level y_{T-1} in the previous period, are given by*

$$\Psi_T(y_{T-1}) = \lambda^{-1}r - \lambda^{-1}\left(c + c\ln\left(\frac{r}{c}\right) + r\lambda y_{T-1}\right)e^{-\lambda y_{T-1}},$$
$$U_T(y_{T-1}) = \lambda^{-1}r - \lambda^{-1}\left(r + r\lambda y_{T-1}\right)e^{-\lambda y_{T-1}}.$$

Then the optimal contract in the second-last period can be determined as follows.

Proposition 3 *If the beginning inventory in the second-last period is $x_{T-1} = (y_{T-2} - D_{T-2})^+$ for a given $y_{T-2} \geq 0$, the optimal order plan in this period is:*

$$q_{T-1}(x_{T-1}) = \begin{cases} y_{T-1}^*, & \textit{if } x_{T-1} = 0, \\ x_{T-1}^* - x_{T-1}, & \textit{if } h < \delta(c + c\ln(\frac{r}{c})) - c \textit{ and } x_{T-1} \in (0, x_{T-1}^*], \\ 0, & \textit{otherwise}, \end{cases}$$

where $x^*_{T-1} = \lambda^{-1} \ln\left(\frac{\delta c}{c+h}\left[1+\ln\left(\frac{r}{c}\right)\right]\right)$ *and* y^*_{T-1} *solves* $(c+h)e^{\lambda y} - \delta\lambda r y = (1-\delta)r + \delta c\left(1+\ln\left(\frac{r}{c}\right)\right) + h$. *Furthermore,* $x^*_{T-1} < y^*_{T-1}$.

This proposition suggests that the optimal contract in the second-last period is more complex than the one in the last period and consists of the following pieces: ordering the amount y^*_{T-1} when $x_{T-1} = 0$, up to a level $x^*_{T-1} (< y^*_{T-1})$ when $x_{T-1} \in (0, x^*_{T-1}]$, and nothing when $x_{T-1} > x^*_{T-1}$. Similar to the static case, the discontinuity of the optimal contract at $x_{T-1} = 0$ is again caused by the point mass at $x_{T-1} = 0$ and the different forms of the virtual surplus $J_{T-1}(y_{T-1} \mid x_{T-1})$ around $x_{T-1} = 0$. This contract is similar to a BOC if x^*_{T-1} is close to zero or a base-stock policy if x^*_{T-1} is close to y^*_{T-1}, depending on the model parameters.

A troubling fact about this contract is that it depends on the exact distribution of x_{T-1}. For instance, if the y_{T-2} in the proposition is a random variable or if x_{T-1} is derived from $(y_{T-3} - D_{T-3} - D_{T-2})^+$, $(y_{T-4} - D_{T-4} - D_{T-3} - D_{T-2})^+$, and so on, the optimal contract will be even more complex. Thus, the optimal contract in the second-last period is generally complicated and history dependent, even under a special demand distribution like exponential. This is somewhat discouraging, and echoes the observation from the existing literature that optimal short-term contracts for two-period problems are often difficult to characterize (see §2).

However, Proposition 3 also implies that if x_{T-1} is derived from $(y_{T-2} - D_{T-2})^+$ and if $h \geq \delta(c + c\ln(\frac{r}{c})) - c$, the middle part of the optimal contract will disappear and the contract will reduce to a simple batch-order contract. In fact, this result does not depend upon the assumption $x_{T-1} = (y_{T-2} - D_{T-2})^+$, as the next proposition shows.

Proposition 4 *If* $h \geq \delta(1+\ln(\frac{r}{c}))c - c$, *the optimal order plan in the second-last period is given by*

$$q_{T-1}(x_{T-1}) = \begin{cases} y^*_{T-1}, & x_{T-1} = 0, \\ 0, & x_{T-1} > 0, \end{cases}$$

where y^*_{T-1} *solves* $(c+h)e^{\lambda y} - \delta\lambda r y = (1-\delta)r + \delta c\left(1+\ln\left(\frac{r}{c}\right)\right) + h$.

Intuitively, high costs diminish the benefit of orders, which will eventually eliminate the orders at all positive x_{T-1} when the costs are high enough. The order may still be profitable when x_{T-1} is zero thanks to the gap between y^*_{T-1} and x^*_{T-1}, or the downward jump in the order plan around $x_{T-1} = 0$.

With the above mixed results, we conclude our analysis of the finite-horizon setting, because a more thorough investigation may obscure our pursuit of the structure of the optimal contracts.

We refer interested readers to Appendix C, in which we analyze a two-period model under general demand distribution and zero initial inventory. The results there further illustrate that the optimal contracts in a relatively general two-period setting can easily become too complex to carry any interesting structure. This, on the other hand, justifies our emphasis on the elegant batch-order contracts in this paper. We have seen from Propositions 2 and 4 that they can be optimal in the finite-horizon case (in certain cost regimes), and we shall show the same in the infinite-horizon case in §5. In the next subsection, we numerically demonstrate that even outside of the optimality regime, the BOCs still perform very well.

4.5 Performance of the Batch-Order Contracts

The optimal BOC performs surprisingly well even outside of its optimality region. To demonstrate that, we conduct a numerical study below, comparing the performances of the optimal BOC and the optimal contract in the second-last period for a wide range of model parameters. To that end, we first compute the expected channel profit and retailer's profit under the two contracts. Let $a \wedge b$ denote $\min(a, b)$.

Proposition 5 *Suppose the post-order inventory in the third-last period is y_{T-2}. Under the optimal contract defined in Proposition 3, when $h < \delta(c + c\ln(\frac{r}{c})) - c$, the expected profits of the last two periods for the channel and the retailer are given by:*

$$\begin{aligned}\Psi_{T-1}^{Opt}(y_{T-2}) =& (v_{T-1}(y_{T-1}^*) - cy_{T-1}^* + \delta\Psi_T(y_{T-1}^*))e^{-\lambda y_{T-2}} \\ &+ \int_{0^+}^{x_{T-1}^* \wedge y_{T-2}} \left(v_{T-1}(x_{T-1}^*) - c(x_{T-1}^* - x) + \delta\Psi_T(x_{T-1}^*)\right) \lambda e^{-\lambda(y_{T-2}-x)} dx \\ &+ \int_{x_{T-1}^* \wedge y_{T-2}}^{y_{T-2}} \left(v_{T-1}(x) + \delta\Psi_T(x)\right) \lambda e^{-\lambda(y_{T-2}-x)} dx,\end{aligned}$$

$$\begin{aligned}U_{T-1}^{Opt}(y_{T-2}) =& (u_{T-1}(x_{T-1}^* \wedge y_{T-2}) - (x_{T-1}^* \wedge y_{T-2})u'_{T-1}(x_{T-1}^*))e^{-\lambda y_{T-2}} \\ &+ \int_{0^+}^{x_{T-1}^* \wedge y_{T-2}} \left(u_{T-1}(x_{T-1}^* \wedge y_{T-2}) - ((x_{T-1}^* \wedge y_{T-2}) - x)u'_{T-1}(x_{T-1}^*)\right) \lambda e^{-\lambda(y_{T-2}-x)} dx \\ &+ \int_{x_{T-1}^* \wedge y_{T-2}}^{y_{T-2}} \left(v_{T-1}(x) + \delta U_T(x)\right) \lambda e^{-\lambda(y_{T-2}-x)} dx,\end{aligned}$$

where $x_{T-1}^* = \lambda^{-1}\ln\left(\frac{\delta c}{c+h}\left[1 + \ln(\frac{r}{c})\right]\right)$ *and* $u_{T-1}(x_{T-1}) = v_{T-1}(x_{T-1}) + \delta U_T(x_{T-1})$.

Under the BOC defined in Proposition 4 (ignoring the cost condition), the expected two-period profits are:

$$\Psi_{T-1}^{BOC}(y_{T-2}) = (v_{T-1}(y_{T-1}^*) - cy_{T-1}^* + \delta\Psi_T(y_{T-1}^*))e^{-\lambda y_{T-2}} + \int_{0^+}^{y_{T-2}} \left(v_{T-1}(x) + \delta\Psi_T(x)\right) \lambda e^{-\lambda(y_{T-2}-x)} dx,$$

$$U_{T-1}^{BOC}(y_{T-2}) = \int_0^{y_{T-2}} \left(v_{T-1}(x) + \delta U_T(x)\right) \lambda e^{-\lambda(y_{T-2}-x)} dx.$$

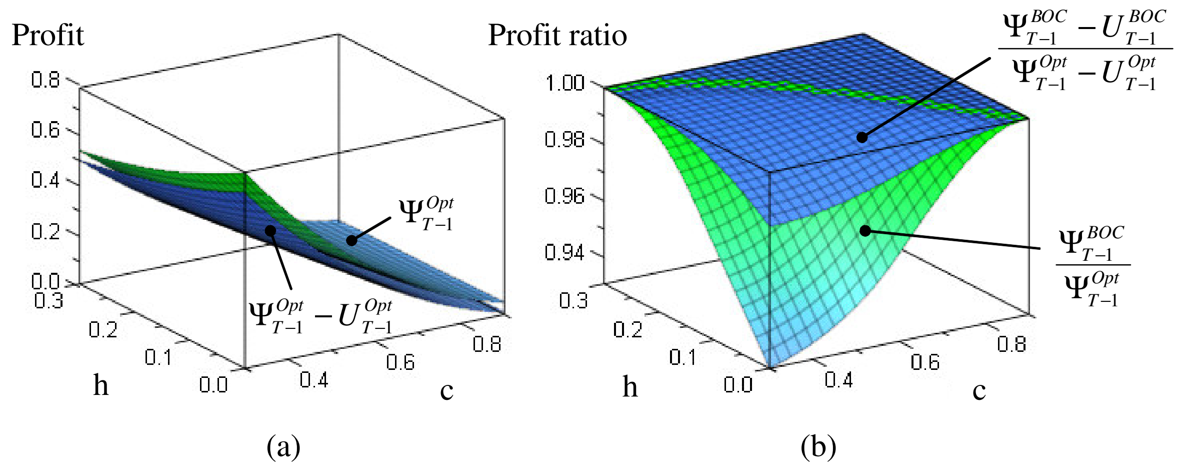

Figure 3: Expected two-period channel profit and supplier's profit (a) under the optimal short-term contract, and (b) under the optimal BOC (relative to the profits under the optimal contract).

When $y_{T-2} > x^*_{T-1}$, the two profit functions under the optimal contract both consist of three parts: the expected profits when $x_{T-1} = 0$, $x_{T-1} \in (0, x^*_{T-1}]$, and $x_{T-1} \in (x^*_{T-1}, y_{T-2}]$; they collapse into two parts when $y_{T-2} \le x^*_{T-1}$. The two profit functions are much simpler under the BOC because the order only occurs at $x_{T-1} = 0$ and the supplier sets the price such that the retailer's expected profit equals her reservation profit $v_{T-1}(0) + \delta U_T(0)$, which is zero.

Without loss of generality, we can normalize the model by selecting $\lambda = 1$ and $r = 1$, because the scaled order plan $\lambda q_t(x_t)$ is solely determined by δ, $\frac{c}{r}$, and $\frac{h}{r}$. We also find that the discount factor δ has a minimum impact on the profit functions as long as it is in a reasonable range, say $[0.8, 1]$, so we set $\delta = 0.9$. Figure 3 exhibits the profits under the two types of contracts for $y_{T-2} = 0.3$, $c \in [0.3, 0.9]$, and $h \in [0, 0.3]$. Panel (a) shows the channel profit and supplier's profit under the optimal contract, i.e., Ψ^{Opt}_{T-1} and $\Psi^{Opt}_{T-1} - U^{Opt}_{T-1}$; panel (b) shows the profits under the optimal BOC relative to those under the optimal contract, i.e., the ratios $\Psi^{BOC}_{T-1} \Big/ \Psi^{Opt}_{T-1}$ and $(\Psi^{BOC}_{T-1} - U^{BOC}_{T-1}) \Big/ (\Psi^{Opt}_{T-1} - U^{Opt}_{T-1})$. According to panel (b), in the given range of parameters the supplier's profit under the BOC is at least 98% of that under the optimal contract. We find this result quite robust against the choice of y_{T-2}. In panel (b), the two ratio functions coincide at the top when the costs fall in the optimality region of the BOC, i.e., $h \ge \delta(c + c\ln(\frac{r}{c})) - c$. Thus, we can reach the conclusion that the optimal BOC performs extremely well compared with the optimal contract even when it is sub-optimal.

The numerical study provides a strong support for studying the BOC. Its ease of analysis and implementation should more than compensate the few percentage loss of the supplier's profit (when it is sub-optimal). Later, in §5, we will conduct another numerical study to demonstrate the effectiveness of the BOC in the infinite-horizon setting as well.

5 Infinite-Horizon Model with Exponential Demand and High Costs

The analyses of the static contracts in §3 and two-period contracts in §4 (and Appendix C) manifest the complexity of the actual expressions for the optimal contracts in the finite-horizon case. Thus, it seems from the outset that there is little hope to derive closed-form expressions for the optimal contract in the infinite-horizon model. With this in mind, our analysis takes the following approach.

We consider an exponential distribution for the demand. For this distribution, in the so-called high-cost regime, we are able to derive in closed form the optimal short-term contract for the supplier, which turns out to be a stationary BOC described by a quantity and payment pair (b^*, s^*). Thus, throughout this section, the name "batch-order contract (BOC)" normally refers to an infinite sequence of identical single-period BOCs, when it is clear from the context. As mentioned in the introduction, the exponential demand is not a bad approximation to what we see in practice in many situations, and is used in the literature to analyze intractable problems. The question, then, remains as to whether the elegant contract structure we derive is useful in situations in which we do not have a provable optimal structure. To answer this question, we test the performance of the best BOC in (i) various cost domains and (ii) across several demand distributions. Through numerical and simulation exercises, we find that the BOCs perform quite well in most of these settings. Though we are unable to prove that the BOCs are optimal for arbitrary demand distributions (except in the exponential case), we think that our analysis here may actually be quite robust. All these reasons, we believe, make a strong case for studying BOCs in settings such as ours, in which downstream inventory information is hidden.

Finally, we note that our philosophy here resonates with the approach used in attacking several problems of interest to operations management researchers. A common tactic in the literature on call center management, where calculating optimal scheduling policies is often infeasible, is to obtain analytical results by using certain assumptions on the inputs in heavy traffic regimes. These results are used to suggest policies in more practical settings, in which their performance is shown empirically to be close to optimal. Another emerging area is in inventory management, where asymptotically optimal results (in the sense of service levels) suggest policies that perform well in non-asymptotic domains.

5.1 Optimal Contract

Our analysis of the optimal contract proceeds as follows. We first prepare the background by computing the expected profit-to-go functions under BOCs. Next, we find the optimal BOC (the batch size b^* and batch payment s^* that maximize the supplier's expected profit-to-go). Finally, we show that when the model parameters lie in a critical region, the contract we obtain is the optimal short-term contract. All proofs are given in Appendix B.

As in the finite-horizon case (§4.4), the retailer's expected revenue in period t from the post-order inventory y_t is given by

$$v(y_t) = \lambda^{-1}(r+h)(1-e^{-\lambda y_t}) - hy_t,$$

taking the end-of-the-period holding cost into account. Its first derivative is

$$v'(y_t) = (r+h)e^{-\lambda y_t} - h.$$

Notice that $v(\cdot)$ and $v'(\cdot)$ are independent of t. Further, if the post-order inventory in period $t-1$ is y_{t-1}, the beginning inventory x_t in period t is distributed according to the following c.d.f. $G_t(\cdot \mid y_{t-1})$ and p.d.f. $g_t(\cdot \mid y_{t-1})$:

$$\begin{aligned} G_t(x_t \mid y_{t-1}) &= e^{-\lambda(y_{t-1}-x_t)}, \quad && x_t \in [0, y_{t-1}], \\ g_t(x_t \mid y_{t-1}) &= \lambda e^{-\lambda(y_{t-1}-x_t)}, && x_t \in (0, y_{t-1}]. \end{aligned}$$

As the first step towards finding the optimal BOC, we compute the expected profit-to-go functions under BOCs. Consider an arbitrary BOC (b, s) from period t onward, regardless of the perceived post-order inventory $\widehat{y}_{t-1}$ in the previous period. The retailer's expected profit-to-go $U_t(y_{t-1} \mid \widehat{y}_{t-1})$ simplifies to $U_t(y_{t-1})$, which can be computed recursively:

$$U_t(y_{t-1}) = \int_{0^+}^{y_{t-1}} \{v(x_t) + \delta U_{t+1}(x_t)\}\, dG_t(x_t \mid y_{t-1}) + \{v(b) - s + \delta U_{t+1}(b)\}\, G_t(0 \mid y_{t-1}). \quad (15)$$

The first part of the expression is the retailer's expected profit-to-go when $x_t > 0$ (with no order in period t) and the second part is that when $x_t = 0$ (with an order of b units). Over an infinite horizon, periods t and $t+1$ are identical and the functions $U_t(\cdot)$ and $U_{t+1}(\cdot)$ can both be replaced by $U_\infty(\cdot)$ (the subscript "∞" means an infinite number of periods to go, contrary to the meaning of the subscript "t"). Thus, equation (15) becomes

$$U_\infty(y) = \int_{0^+}^{y} \{v(x) + \delta U_\infty(x)\} \lambda e^{-\lambda(y-x)} dx + \{v(b) - s + \delta U_\infty(b)\} e^{-\lambda y}. \tag{16}$$

Note that, in the expression, y is the post-order inventory in the "previous" period, while x is the beginning inventory of the "current" period.

Similarly, the retailer's default profit-to-go given previous period's inventory y_{t-1} (with no future orders) can be determined recursively as

$$\begin{aligned}\underline{U}_t(y_{t-1}) &= \int_{0}^{y_{t-1}} \left\{v(x_t) + \delta \underline{U}_{t+1}(x_t)\right\} dG_t(x_t \mid y_{t-1}) \\ &= \int_{0^+}^{y_{t-1}} \left\{v(x_t) + \delta \underline{U}_{t+1}(x_t)\right\} \lambda e^{-\lambda(y_{t-1}-x_t)} dx_t + 0 \cdot e^{-\lambda y_{t-1}},\end{aligned}$$

which, over the infinite horizon, becomes

$$\underline{U}_\infty(y) = \int_{0^+}^{y} \{v(x) + \delta \underline{U}_\infty(x)\} \lambda e^{-\lambda(y-x)} dx.$$

Finally, the supplier's expected profit-to-go given the post-order inventory y_{t-1} in period $t-1$ can be expressed recursively as

$$\Pi_t(y_{t-1}) = \int_{0^+}^{y_{t-1}} \delta \Pi_{t+1}(x_t) dG_t(x_t \mid y_{t-1}) + \{s - cb + \delta \Pi_{t+1}(b)\} G_t(0 \mid y_{t-1}),$$

which simplifies to

$$\Pi_\infty(y) = \int_{0^+}^{y} \delta \Pi_\infty(x) \lambda e^{-\lambda(y-x)} dx + \{s - cb + \delta \Pi_\infty(b)\} e^{-\lambda y} \tag{17}$$

over the infinite horizon. The expected profit-to-go for the channel is simply $\Psi_\infty(y) = U_\infty(y) + \Pi_\infty(y)$.

Through the transformation $\widetilde{h}(y) = e^{\lambda y} h(y)$, the above recursive expressions can be transformed into ordinary differential equations, which can be solved in closed form.

Proposition 6 *Under a BOC (b, s) and exponential demand with rate λ, given post-order inventory y of the previous period, the expected profits-to-go for the retailer, the supplier and the channel are given by:*

$$U_\infty(y) = \omega(y) + M_u e^{-\lambda(1-\delta)y}, \qquad \textit{with} \;\; M_u = -\frac{\frac{r}{\delta\lambda} + \frac{h}{\delta(1-\delta)\lambda} + \frac{h}{1-\delta} b + s}{1 - \delta e^{-\lambda(1-\delta)b}} < 0, \tag{18}$$

$$\underline{U}_\infty(y) = \omega(y) + \underline{M}_u e^{-\lambda(1-\delta)y}, \qquad \textit{with} \;\; \underline{M}_u = -\frac{(1-\delta)r + h}{\delta(1-\delta)^2\lambda} < 0, \tag{19}$$

$$\Pi_\infty(y) = M_\pi e^{-\lambda(1-\delta)y}, \qquad \textit{with} \;\; M_\pi = \frac{s - cb}{1 - \delta e^{-\lambda(1-\delta)b}} > 0 \tag{20}$$

$$\Psi_\infty(y) = \omega(y) + M_\psi e^{-\lambda(1-\delta)y}, \qquad \textit{with} \;\; M_\psi = -\frac{\frac{r}{\delta\lambda} + \frac{h}{\delta(1-\delta)\lambda} + \frac{h}{1-\delta} b + cb}{1 - \delta e^{-\lambda(1-\delta)b}} < 0, \tag{21}$$

where

$$\omega(y) = -\frac{h}{1-\delta}y + \frac{(1-\delta)r + (2-\delta)h}{(1-\delta)^2\lambda} + \frac{r+h}{\delta\lambda}e^{-\lambda y}. \tag{22}$$

Next, the best batch-order quantity b^* and payment s^* for the supplier can be uniquely determined.

Theorem 4 *The optimal BOC (b^*, s^*) for the supplier is determined by:*

$$e^{\lambda(1-\delta)b^*} - \delta(1-\delta)\lambda b^* = 1 + \frac{(1-\delta)^2(r-c)}{h+(1-\delta)c}, \tag{23a}$$

$$\textit{and} \quad \frac{\frac{r}{\delta\lambda} + \frac{h}{\delta(1-\delta)\lambda} + \frac{h}{1-\delta}b^* + s^*}{1-\delta e^{-\lambda(1-\delta)b^*}} = \frac{(1-\delta)r + h}{\delta(1-\delta)^2\lambda}. \tag{23b}$$

Under this contract, the retailer only earns her reservation profits, i.e., $U_\infty(y) = \underline{U}_\infty(y)$ for all $y \geq 0$.

Under the above BOC, the retailer only orders when her beginning inventory hits zero. The supplier can set the price s^* high enough to make the retailer break even and earn her reservation profit when she places the order. When the beginning inventory is positive, the retailer places no order and only enjoys her reservation profit. Thus, under any circumstances, the retailer makes exactly the reservation profit, or, in order words, the information rent is always zero. This is an important reason the optimal BOC may be particularly attractive to the supplier – it minimizes the retailer's information advantage and maximizes the supplier's leverage in splitting the channel profits. On the other hand, the BOC is drastically different from the first-best policy (a base-stock policy), which may result in severe loss of channel efficiency. This trade-off between information rent and system efficiency is the main trade-off faced by the supplier, as in any other adverse selection problems.

Our last step involves showing that the optimal BOC derived above is optimal at least when the cost/price ratios, $\frac{c}{r}$ and $\frac{h}{r}$, are high. Proposition 1 presents a necessary condition for the period-t post-order inventory $y_t(x_t)$ to be optimal, which suggests the examination of the term $\frac{\partial J_t(y_t \mid x_t)}{\partial y_t}$. The optimal batch size b^* found above is in fact the optimal $y_t(0)$, which indeed satisfies $\frac{\partial J_t(b^* \mid 0)}{\partial y_t} = 0$ and the corresponding second-order condition (SOC). A main step in our analysis is to verify that, for $x_t > 0$, the optimal $y_t(x_t) = x_t$, i.e., order nothing. To that end, it suffices to show $\frac{\partial J_t(y_t \mid x_t)}{\partial y_t} < 0$ for all $y_t \geq x_t > 0$. In general, this is a daunting task because the virtual surplus $J_t(y_t \mid x_t)$ depends upon the supplier's belief G_t, which is updated from G_{t-1}, G_{t-2}, and so on, through Bayes' rule. The belief process is typically extremely complicated and depends upon the entire history of the contracts and the retailer's orders. However, under exponential demand, the belief process has a strong property that $\frac{G_t(x_t)}{g_t(x_t)} \geq \lambda^{-1}$ for $x_t \in (0, \overline{x}_t]$, for some upper bound $\overline{x}_t$ (Theorem 3). This property leads to the following sufficient condition for the optimality of $y_t(x_t) = x_t$ when $x_t > 0$.

Lemma 2 *Suppose $t \geq 2$. If a BOC (b, s) is offered from period $t+1$ onward and $\delta\lambda b \leq 1$, then in period t, $\frac{\partial J_t(y_t \mid x_t)}{\partial y_t} < 0$ for all $y_t \geq x_t > 0$.*

We exclude the case $t = 1$ from the lemma, because G_1 is a special distribution with $G_1(0) = 1$ and hence it is unnecessary to consider $x_1 > 0$. Ultimately, the simple condition $\delta\lambda b^* \leq 1$ leads to a cost region where the optimal BOC is the optimal short-term contract. Define the *relative production cost* $\widetilde{c} = \frac{c}{r}$ and *relative holding cost* $\widetilde{h} = \frac{h}{r}$. We now present the main result of this section.

Theorem 5 *Suppose the demand is exponential and i.i.d. in every period and the relative costs $\widetilde{c}$ and $\widetilde{h}$ lie above the following line:*

$$\left[e^{\frac{1-\delta}{\delta}} + \delta - 2\right] \widetilde{h} + (1-\delta)\left[e^{\frac{1-\delta}{\delta}} - 1\right] \widetilde{c} = (1-\delta)^2. \tag{24}$$

Then, for any $K \geq 2$, if the optimal BOC (b^*, s^*) defined by (23) is applied from period K onward, the optimal contracts in periods 1 through $K-1$ are also the (b^*, s^*) contract.

The theorem suggests that if the supplier will ever offer the (b^*, s^*) contract in the future, it should be offered from the start. Because K can be arbitrarily large, the assumption about the future offer is negligible although it cannot be completely assumed away. This is a strong result that prevents deviations over any finite number of periods.

Figure 4 illustrates the boundary line of the high-cost region in which the BOC (b^*, s^*) is optimal, for different values of the discount factor δ. As the figure shows, the larger δ is the higher the costs should be. To get an idea of the type of products that are described by this cost regime, if we rewrite the discount factor as a function of interest rate and impute the per-period holding cost as being close to the per-period interest on the production cost c, we see that slow-moving items with relatively low service levels fit the description. We note, however, that our numerical results show that for different demand distributions the BOC performs very well for a larger class of products (including ones with substantially higher service levels) that are not described theoretically.

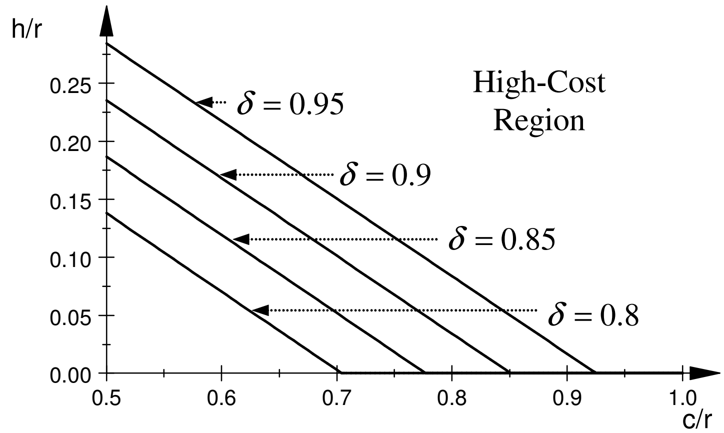

Figure 4: The region of relative costs for the optimal BOC to be the optimal short-term contract, given δ.

5.2 Performance of the Batch-Order Contracts

In this subsection, we test the performance of the optimal BOC in various scenarios. Clearly, all our assumptions that lead to the optimality result need to be relaxed and the performance thereof tested. It is interesting and important to understand the loss to the system due to information asymmetry even when the BOCs are optimal (i.e., in the high-cost domains with exponential demand). This gives us an idea of the value of information. We also compare the BOC to simple wholesale-price contracts (where a unit price w_t is used in period t). We then check the performance of the contract derived by optimizing over the class of BOCs in situations in which the BOC may not be optimal. These include situations in which the costs are not in the high-cost domain and when the demand distribution is not necessarily exponential.

In the single-period case, we have performed extensive numerical experiments using various combinations of demand and initial-inventory distributions, which are omitted here due to space limitations. A few noteworthy facts emerge from this analysis. For any distribution of the demand, the supplier's profits are at the lowest when the initial-inventory distribution is uniform. In the worst possible case, our numerical results indicate that the optimal BOC brings about an increase in profits that is significantly greater than when optimizing over classes of certain simpler contracts such as linear or piecewise-linear contracts. We have presented some numerical results for the two-period model under exponential demand in §4.5. The results show that the BOC performs extremely well compared with the optimal contract, for a wide range of parameters.

For the infinite-horizon case, we begin with a numerical comparison of the optimal BOC with the optimal wholesale-price contract (WPC) and the first-best contract (FBC) under exponential demand distribution. We note that the optimal WPC and the FBC are both stationary base-stock policies, which can be computed relatively easily, and hence their derivations are omitted. We shall focus on the channel profits and the supplier's profits when the system starts with zero inventory, i.e., $\Psi_\infty(0)$ and $\Pi_\infty(0)$. For brevity and clarity, we denote the profits by symbols such as Ψ_{BOC} and Π_{BOC}, where the subscript represents the contract type and the (zero) initial inventory is implicit. Assume $\delta = 0.9$, $\lambda = 0.1$, $0.4 \leq c/r \leq 0.9$, and $0 \leq h/r \leq 0.3$; Figure 5 illustrates the expected channel profits and supplier's profits under the three contracts, in both absolute and relative terms (panels a and b, respectively). Under an optimal BOC, when the beginning inventory is zero, the supplier can extract all channel profit (by Theorem 4 and the fact that $\underline{U}_\infty(0) = 0$), and therefore $\Psi_{BOC} = \Pi_{BOC}$. The same is true for the FBC and hence $\Psi_{FBC} = \Pi_{FBC}$. Thus, there are only four distinctive absolute-profit measures, Ψ_{FBC}, Ψ_{BOC}, Ψ_{WPC} and Π_{WPC}, which are depicted in Figure 5(a). Figure 5(b) depicts three relative-profit measures:

$$\frac{\Psi_{BOC}}{\Psi_{FBC}}, \quad \frac{\Psi_{WPC}}{\Psi_{FBC}}, \text{ and } \frac{\Pi_{WPC}}{\Pi_{FBC}}.$$

Notice that $\frac{\Pi_{BOC}}{\Pi_{FBC}} = \frac{\Psi_{BOC}}{\Psi_{FBC}}$. Another interesting measure, $\frac{\Pi_{WPC}}{\Pi_{BOC}}$, can be simply obtained from $\frac{\Pi_{WPC}}{\Pi_{FBC}} \Big/ \frac{\Psi_{BOC}}{\Psi_{FBC}}$.

Obviously, as the relative costs c/r and h/r increase, the expected channel profits and supplier's profits under all contracts decline. A closer examination of Figure 5(b) tells that: (1) the channel profit (supplier's profit) under the optimal BOC captures 85% - 95% of the first-best channel profit (supplier's profit); (2) the channel profit under the optimal WPC captures 76% - 81% of the first-best channel profit; and (3) the supplier's profit under the optimal WPC only captures 52% - 63% of his first-best profit. Therefore, the BOC performs very well against the FBC and significantly better than the WPC across the given cost region. The performance enhances as the relative costs increase (and moves into the optimality region of the BOC). If the relative costs are low, the BOC may no longer be optimal but is still a good heuristic solution. Because it already captures at least 85% of the first-best profit, the benefit of a more sophisticated contract in order to achieve supplier optimality seems not very substantial (at the boundary of the high-cost region, even the optimal short-term contract can only capture about 86.6% of the first-best profit). These observations are consistent with the two-period results presented in §4.5.

Next, we expand this study to a wider range of model parameters and to other demand distributions through simulation. An interesting numerical inference we observe from our experiments

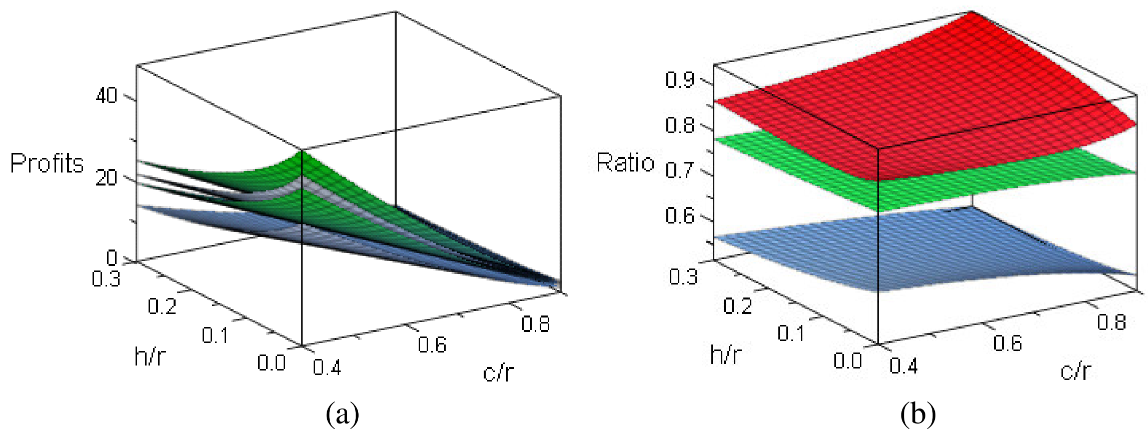

Figure 5: (a) Expected channel profits and supplier's profits under the three contracts: Ψ_{FBC} (Π_{FBC}), Ψ_{BOC} (Π_{BOC}), Ψ_{WPC}, and Π_{WPC}, in decreasing order. (b) Profit ratios Ψ_{BOC}/Ψ_{FBC}, Ψ_{WPC}/Ψ_{FBC}, and Π_{WPC}/Π_{FBC}, in decreasing order. Assume $\delta = 0.9$, $0.4 \leq c/r \leq 0.9$, $0 \leq h/r \leq 0.3$, and exponential demand with rate $\lambda = 0.1$.

is that in high-cost domains, the optimal BOC for several often-used distributions performs quite well when compared to the first-best. This leads us to think that it is plausible that BOCs are either optimal or close to optimal for a wider set of scenarios than what we have characterized in our paper, although we are unable to prove this conjecture analytically.

We use the following relative performance benchmarks: $\frac{\Psi_{BOC}}{\Psi_{FBC}}$, $\frac{\Psi_{WPC}}{\Psi_{FBC}}$, and $\frac{\Pi_{WPC}}{\Pi_{BOC}}$. The other interesting measure, $\frac{\Pi_{WPC}}{\Pi_{FBC}}$, can be obtained through $\frac{\Pi_{WPC}}{\Pi_{BOC}} \times \frac{\Psi_{BOC}}{\Psi_{FBC}}$. We report results for four distributions: Exponential, Uniform, Normal, and Gamma. The performance benchmarks were quite robust to the specific parameters of these distributions and the discount factor. Thus, we do not report these. However, the cost parameters are more important. We look at values $\frac{c}{r} \in (0, 1)$ and $\frac{h}{r} \in (0, 0.5)$, which we believe covers all realistic scenarios and includes both the high-cost domain as well as cost regions where we may not have BOCs as being optimal.

As can be seen from the table, the general trend is that the use of the optimal BOC on average results in less than 15% of loss as compared to the first-best situation. This is noteworthy in that we are looking at a system in which the supplier is unable to observe downstream inventory information over the entire horizon. Thus, the value of optimal dynamic contracting seems to be quite high. This is especially the case when we compare with simpler mechanisms such as wholesale-price contracts, in which the loss is considerable, especially for the supplier. We do not report here the results obtained when we used a slightly more complicated quantity discount schedule (with

Measure	Ψ_{BOC}/Ψ_{FBC}		Ψ_{WPC}/Ψ_{FBC}		Π_{WPC}/Π_{BOC}	
	Mean	St.dev.	Mean	St.dev.	Mean	St.dev.
Distribution	%	%	%	%	%	%
Exponential	91	6.7	79	8.2	53	8.5
Uniform	86	7.3	74	5.3	46	4.0
Truncated Normal	90	5.8	72	4.6	58	8.0
Gamma	90	5.8	81	5.9	48	4.7

Table 1: Performance of the BOC: the channel-profit ratio between the optimal BOC and the FBC, channel-profit ratio between the optimal WPC and the FBC, and supplier-profit ratio between the optimal WPC and optimal BOC.

a few linear pieces). We find that quantity discount schedules in general perform much better than simple wholesale-price mechanism. This should not come as a surprise, as the optimal static contract displayed a quantity discount property (with infinite number of pieces). However, when we extend the length of the horizon, BOCs dominate simple quantity discount schedules as well. In summary, the BOC performs quite well as compared to first-best and much better than often used simpler contracts.

As mentioned, the results are robust to discount factors, though the general trend indicates that BOCs perform slightly better when δ is smaller. As the relative costs increase, the performance of the BOC for the supplier becomes more significant, though obviously the size of the total pie gets smaller.

As a final remark of this section, we note that the mode of contracting has a significant impact on the supplier's profit. If the system starts with zero inventory, the supplier can implement the first-best order plan and extract all channel profit through a long-term contract: he can simply charge a unit price c in every period, to induce first-best actions from the retailer, and a lump-sum fee at the beginning of the horizon, to extract the channel profit. This contract is feasible only if the supplier can make a credible commitment that he will not raise the price later or take the money and run. Further, it can extract all channel profit only if the initial inventory is public information. The above numerical and simulation results show that the inability to carry out a long-term contract costs the supplier 5% - 15% of the potential profit. Although this may be viewed as an argument in favor of long-term contracts, we must keep in mind that the type of contract which can actually be used is often dictated by real-world conditions and that short-term contracts are widely used in supply chain environments.

6 Conclusion

We have analyzed a dynamic adverse selection model in which a supplier sells to a downstream retailer. Our analysis yields insights that are potentially valuable to academics and practitioners. First, we demonstrate that information asymmetry has a clear and negative impact on system efficiency. In the single-period case, the supplier's optimal contract structure is such that when the retailer reports a high initial inventory, the contract deems the situation as unfavorable to the supplier (because the retailer's purchasing requirement is small) and there will be no trade. Though this seems somewhat extreme, the obvious intuition is that the optimal contract ensures that the retailer will not inflate her inventory position to secure deep discounts from the supplier. Clearly, due to this distortion effect, the system loses as compared to the first-best benchmark. Our numerical analysis suggests that the optimal contract captures significantly greater profits than simpler contracts such as linear or simple piecewise-linear contracts. Similar results are shown in the two-period case, which demonstrate the value of optimal contracting under asymmetric inventory information in the finite-horizon setting.

When we analyze the infinite-horizon contract with exponential demand, we show that the optimal contract in the high cost region is a stationary batch-order contract. A BOC is reminiscent of the well known (s, S) policy in inventory systems with fixed costs. Thus, one insight that we get is that the effect of asymmetric information imputes a fixed charge to this supply chain. While the relationship between this charge and the exact value of information is not obvious, the above structure is nevertheless somewhat interesting. Further, the BOC is of value to the supplier even when it is not optimal. Extensive numerical experiments (in the exponential demand case and otherwise) indicate that optimizing over BOCs dominates using simple well-understood contracts such as linear and piecewise-linear wholesale-price contracts. An important feature of BOCs is their elegant form and ease of implementation. The fact is particularly striking in a problem like this when even the optimal two-period contract can be extremely complex.

In our analysis, we have made several assumptions, some of which are somewhat strong. However, given the difficulty of the analysis and the fact that this is the first piece of work in this area, we feel that the assumptions may be warranted. Further, we are able to obtain sharp results for this problem once these assumptions are made. As we have seen, the results by themselves seem robust when we check them on various scenarios in which our assumptions may not hold, which leads us to believe that they may have significant practical value. Further, we hope that the progress we have made on a challenging problem encourages future research in this area.

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Appendix A - Solving the Single-Period Model

First, recall that

$$v_1(y_1) = rE\min\{y_1, D_1\} = \int_0^{y_1} r\overline{F}(\xi)d\xi, \tag{A1a}$$

$$v_1'(y_1) = r\overline{F}(y_1) \geq 0, \tag{A1b}$$

$$v_1''(y_1) = -rf(y_1) \leq 0. \tag{A1c}$$

Because $u_1'(x_1) = v_1'(x_1 + q_1(x_1))$, we can show that $u_1(y_0) = v_1(y_0)$ and

$$u_1(x_1) = u_1(y_0) - \int_{x_1}^{y_0} v_1'(\widehat{x}_1 + q_1(\widehat{x}_1))d\widehat{x}_1, \ x_1 \in [0, y_0].$$

In order to prove Theorems 1 and 2, we first show the following lemma.

Lemma A1 *1. An incentive compatible order plan $q_1(x_1)$ must be (weakly) decreasing in x_1.*

2. At an optimal solution, the IR constraint (1c) must be binding at y_0 and redundant at $x_1 \in [0, y_0)$.

3. A contract $\{s_1(x_1), q_1(x_1)\}$ satisfies the global IC constraint (1b) if and only if (4) holds and $q_1(x_1)$ is weakly decreasing in x_1.

Proof of Lemma A1. (1) In the single-period setting, if there exist $x_1' \neq x_1''$ such that $q_1(x_1') = q_1(x_1'')$, then the contract must ensure $s_1(x_1') = s_1(x_1'')$; otherwise, the retailer facing inventory x_1' (x_1'') will report x_1'' (x_1') when $s_1(x_1') < (>) s_1(x_1'')$ and incentive compatibility is violated. Thus, any possible $q_1(x_1)$ is paired with a unique $s_1(x_1)$ and the revelation contract $\{(s_1(x_1), q_1(x_1)\}$ is equivalent to a tariff $\{s_1(q_1)\}$. Facing a tariff, the retailer will choose $q_1(x_1) = \arg\max_{q_1}\{v_1(x_1 + q_1) - s_1(q_1)\}$. Since $\frac{\partial^2 v_1(x_1+q_1)}{\partial x_1 \partial q_1} = v_1''(x_1 + q_1) \leq 0$, $v_1(x_1 + q_1) - s_1(q_1)$ has decreasing differences and $q_1(x_1)$ must be (weakly) decreasing in x_1 (Topkis 1998).

(2) From (1b), we have $v_1(x_1+q_1(x_1))-s_1(x_1)-v_1(x_1) \geq v_1(x_1+q_1(y_0))-s_1(y_0)-v_1(x_1)$. From $v_1''(y_1) \leq 0$ and part (1), we have $v_1(x_1+q_1(y_0))-v_1(x_1)-s_1(y_0) \geq v_1(y_0+q_1(y_0))-v_1(y_0)-s_1(y_0) \geq 0$. Since the supplier's problem maximizes $s_1(\cdot)$, $v_1(y_0 + q_1(y_0)) - v_1(y_0) - s_1(y_0) = 0$ at optimum.

(3) Consider the "$\Rightarrow$" direction and assume the contract $\{s_1(x_1), q_1(x_1)\}$ satisfies the global IC constraint (1b). Then, the analysis in the main text (the envelope theorem) leads to the local IC constraint (3), and part (1) above proves the (weak) monotonicity of $q_1(x_1)$. From expression (3)

to (4), we need to verify that $u_1(x_1)$ is continuous at 0 (even when $q_1(x_1)$ is discontinuous at 0 in the case of $G(0) > 0$). Consider the IC constraints (1b) involving $x_1 = 0$ or $\widehat{x}_1 = 0$:

$$v_1(x_1 + q_1(x_1)) - s_1(x_1) \geq v_1(x_1 + q_1(0)) - s_1(0), \quad x_1 \in (0, y_0],$$
$$v_1(q_1(0)) - s_1(0) \geq v_1(q_1(x_1)) - s_1(x_1), \quad x_1 \in (0, y_0].$$

They are equivalent to:

$$u_1(x_1) \geq u_1(0) + v_1(x_1 + q_1(0)) - v_1(q_1(0)), \quad x_1 \in (0, y_0], \tag{A2}$$
$$u_1(0) \geq u_1(x_1) + v_1(q_1(x_1)) - v_1(x_1 + q_1(x_1)), \quad x_1 \in (0, y_0]. \tag{A3}$$

By letting $x_1 = 0^+$ in (A2) and (A3), we obtain $0 \geq u_1(0) - u_1(0^+) \geq 0$ by continuity of $v_1(\cdot)$. Thus, $u_1(0) = u_1(0^+)$.

For the reverse direction "$\Leftarrow$", consider the case $x_1 > \widehat{x}_1$ (the case $x_1 < \widehat{x}_1$ is similar). We have

$$\begin{aligned}
&u_1(x_1) - [v_1(x_1 + q_1(\widehat{x}_1)) - s_1(\widehat{x}_1)] \\
&= \left[u_1(y_0) - \int_{x_1}^{y_0} v_1'(z_1 + q_1(z_1))dz_1\right] - [u_1(\widehat{x}_1) + v_1(x_1 + q_1(\widehat{x}_1)) - v_1(\widehat{x}_1 + q_1(\widehat{x}_1))] \\
&= \left[u_1(\widehat{x}_1) + \int_{\widehat{x}_1}^{x_1} v_1'(z_1 + q_1(z_1))dz_1\right] - \left[u_1(\widehat{x}_1) + \int_{\widehat{x}_1}^{x_1} v_1'(z_1 + q_1(\widehat{x}_1))dz_1\right] \\
&= \int_{\widehat{x}_1}^{x_1} \left[v_1'(z_1 + q_1(z_1)) - v_1'(z_1 + q_1(\widehat{x}_1))\right] dz_1.
\end{aligned}$$

Because $q_1(z_1) \leq q_1(\widehat{x}_1)$ and $v_1''(\cdot) \leq 0$, the above expression must be non-negative and the global IC constraint is satisfied. ■

Proof of Theorem 1. Substituting $s_1(x_1) = v_1(x_1 + q_1(x_1)) - u_1(x_1)$ in the objective function (1a) and using (4), we obtain

$$\begin{aligned}
&\int_0^{y_0} \{v_1(x_1 + q_1(x_1)) - u_1(x_1) - cq_1(x_1)\} dG(x_1) \\
&= \int_0^{y_0} \{v_1(x_1 + q_1(x_1)) - cq_1(x_1)\} dG(x_1) + \int_0^{y_0} \int_{x_1}^{y_0} v_1'(\widehat{x}_1 + q_1(\widehat{x}_1)) d\widehat{x}_1 dG(x_1) - u_1(y_0) \\
&= \int_0^{y_0} \{v_1(x_1 + q_1(x_1)) - cq_1(x_1)\} dG(x_1) + \int_0^{y_0} \int_0^{\widehat{x}_1} v_1'(\widehat{x}_1 + q_1(\widehat{x}_1)) dG(x_1) d\widehat{x}_1 - u_1(y_0) \\
&= \int_0^{y_0} \{v_1(x_1 + q_1(x_1)) - cq_1(x_1)\} dG(x_1) + \int_0^{y_0} v_1'(\widehat{x}_1 + q_1(\widehat{x}_1)) G(\widehat{x}_1) d\widehat{x}_1 - u_1(y_0) \\
&= \int_{0^+}^{y_0} \left\{v_1(x_1 + q_1(x_1)) - cq_1(x_1) + v_1'(x_1 + q_1(x_1)) \frac{G(x_1)}{g(x_1)}\right\} g(x_1) dx_1 \\
&\quad + \{v_1(q_1(0)) - cq_1(0)\} G(0) - u_1(y_0) \\
&= \int_{0^+}^{y_0} J_1(x_1, q_1(x_1)) g(x_1) dx_1 + J_1(0, q_1(0)) G(0) - u_1(y_0).
\end{aligned}$$

The above function is maximized by $q_1(x_1) = \arg\max_{q_1 \geq 0} J_1(x_1, q_1(x_1)),\ x_1 \in [0, y_0]$. If this $q_1(x_1)$ is weakly decreasing, Lemma A1(3) implies that it is incentive compatible and thus solves the supplier's problem. Note that for any $x_1 \in (0, y_0]$, the solution to $\max_{q_1 \geq 0} J_1(x_1, q_1)$ is either 0 or solves the FOC $\frac{\partial J_1(x_1,q_1)}{\partial q_1} = 0$. For $x_1 = 0$, the FOC $v_1'(q_1(0)) - c = 0$ leads to $F(q_1(0)) = \frac{r-c}{r}$. If $G(0) = 0$, we have $\lim_{x_1 \to 0^+} G(x_1) = 0$ and $\lim_{x_1 \to 0^+} J_1(x_1, q_1) = v_1(q_1) - cq_1$, and hence $\lim_{x_1 \to 0^+} q_1(x_1) = q_1(0)$. If $G(0) > 0$, because $q_1(0^+) \equiv \lim_{x \to 0^+} q_1(x_1)$ solves $F(q_1(0^+)) + f(q_1(0^+))\frac{G(0^+)}{g(0^+)} = \frac{r-c}{r}$ and $f(q_1(0^+))\frac{G(0^+)}{g(0^+)} > 0$, we must have $F(q_1(0^+)) < \frac{r-c}{r}$ and thus $q_1(0^+) < q_1(0)$. The SOC amounts to

$$\partial^2 J_1(x_1, q_1)/\partial q_1^2 = v_1''(x_1 + q_1) + v_1'''(x_1 + q_1)\frac{G(x_1)}{g(x_1)} \leq 0, \quad \text{or} \quad \frac{g(x_1)}{G(x_1)} + \frac{v_1'''(x_1 + q_1)}{v_1''(x_1 + q_1)} \geq 0.$$

To ensure the above $q_1(x_1)$ decreasing in x_1, it is sufficient to show that

$$\partial^2 J_1(x_1, q_1)/\partial x_1 \partial q_1 = v_1''(x_1 + q_1) + v_1'''(x_1 + q_1)\frac{G(x_1)}{g(x_1)} + v_1''(x_1 + q_1)\frac{d}{dx}\left(\frac{G(x_1)}{g(x_1)}\right) \leq 0$$

(Topkis 1998). The inequality holds when $\frac{d}{dx_1}\left(\frac{G(x_1)}{g(x_1)}\right) \geq 0$. The remaining results follow by substituting expressions for derivatives of $v_1(\cdot)$, (A1b) and (A1c). ■

Proof of Theorem 2. Since $q_1(x_1)$ solves $F(x_1 + q_1) + f(x_1 + q_1)\frac{G(x_1)}{g(x_1)} = \frac{r-c}{r}$ for any $x_1 > 0$, the first-order derivative with respect to x_1 of the left hand side yields:

$$\begin{aligned}
& f(x_1 + q_1)\left(1 + q_1'(x_1)\right) + f'(x_1 + q_1)\frac{G(x_1)}{g(x_1)}\left(1 + q_1'(x_1)\right) + f(x_1 + q_1)\frac{d}{dx_1}\left(\frac{G(x_1)}{g(x_1)}\right) \\
&= \left(f(x_1 + q_1) + f'(x_1 + q_1)\frac{G(x_1)}{g(x_1)}\right)\left(1 + q_1'(x_1)\right) + f(x_1 + q_1)\frac{d}{dx_1}\left(\frac{G(x_1)}{g(x_1)}\right) \\
&= f(x_1 + q_1)\frac{G(x_1)}{g(x_1)}\left(\frac{g(x_1)}{G(x_1)} + \frac{f'(x_1 + q_1)}{f(x_1 + q_1)}\right)\left(1 + q_1'(x_1)\right) + f(x_1 + q_1)\frac{d}{dx_1}\left(\frac{G(x_1)}{g(x_1)}\right).
\end{aligned}$$

We know that $\frac{g(x_1)}{G(x_1)} + \frac{f'(x_1+q_1)}{f(x_1+q_1)} \geq 0$ and $\frac{d}{dx_1}\left(\frac{G(x_1)}{g(x_1)}\right) \geq 0$ from conditions (6) and (7). Further, if the first inequality is strict, we must have $1 + q_1'(x_1) \leq 0$ for the above expression to equal zero, and hence $q_1'(x_1) \leq -1$.

The functions $q_1(x_1)$, $v_1(x_1)$, $u_1(x_1)$ and $s_1(x_1)$ are continuous for $x_1 > 0$. By equation (2), $s_1(x_1) = v_1(x_1 + q_1(x_1)) - u_1(x_1)$, and by equation (3),

$$s_1'(x_1) = v_1'(x_1 + q_1(x_1))(1 + q_1'(x_1)) - v_1'(x_1 + q_1(x_1)) = v_1'(x_1 + q_1(x_1))q_1'(x_1).$$

Equation (A1b) implies

$$\frac{ds_1}{dq_1} = \frac{s_1'(x_1)}{q_1'(x_1)} = r\overline{F}(x_1 + q_1(x_1)) \geq 0.$$

Furthermore,

$$\begin{aligned}
\frac{d^2 s_1}{(dq_1)^2} &= \frac{d}{dx_1}\left(\frac{ds_1}{dq_1}\right)\Big/ q_1'(x_1) \\
&= \frac{-rf(x_1+q_1(x_1))\,(1+q_1'(x_1))}{q_1'(x_1)} \\
&= -rf(x_1+q_1(x_1))\left((q_1'(x_1))^{-1}+1\right).
\end{aligned}$$

Because $(q_1'(x_1))^{-1} \geq -1$ from part (1), $\frac{d^2 s_1}{(dq_1)^2} \leq 0$. ■

Exponential demand distribution

We now look at some special distributions. We first consider exponential demand with rate λ. The demand distribution has c.d.f. $F(\xi) = 1 - e^{-\lambda\xi}$ and p.d.f. $f(\xi) = \lambda e^{-\lambda\xi}$, $\xi \geq 0$. This distribution has the following properties ($a \wedge b$ denotes $\min(a, b)$):

$$\begin{aligned}
F(\xi) &= 1 - e^{-\lambda\xi},\ \xi \geq 0, \\
f(\xi) &= \lambda e^{-\lambda\xi},\ \xi \geq 0, \\
f'(\xi) &= -\lambda^2 e^{-\lambda\xi} = -\lambda f(\xi),\ \xi \geq 0, \\
E(y \wedge D) &= \lambda^{-1}(1 - e^{-\lambda y}), \quad y \geq 0.
\end{aligned}$$

Under the assumptions of lost sales and no salvage value for unsold items, according to (A1), the retailer's revenue function and its derivatives are given by: $v_1(y_1) = \lambda^{-1} r(1 - e^{-\lambda y_1})$, $v_1'(y_1) = re^{-\lambda y_1}$, and $v_1''(y_1) = -r\lambda e^{-\lambda y_1}$, and the first-order derivative $\frac{\partial J_1(x_1,q_1)}{\partial q_1}$ is given by (9).

Consider two cases of the initial-inventory distribution $G(x_1)$. First, assume $G(x_1)$ is uniform. Since $\frac{G(x_1)}{g(x_1)} = x_1$, the FOC (8) reduces to $r\,(1-\lambda x_1)\,e^{-\lambda(x_1+q_1)} - c = 0$. The optimal order plan $q_1(x_1)$ can be computed from Theorem 1 and equation (8). Notice that there is a unique $\overline{x}_1 < \lambda^{-1}$ such that $r\,(1-\lambda\overline{x}_1)\,e^{-\lambda\overline{x}_1} - c = 0$, or, $\lambda^{-1}\ln\left[\frac{r}{c}\,(1-\lambda\overline{x}_1)\right] - \overline{x}_1 = 0$. For $x_1 \leq \overline{x}_1$, the conditions (6) and (7) hold true: $\frac{g(x_1)}{G(x_1)} + \frac{f'(x_1+q_1)}{f(x_1+q_1)} = x_1^{-1} - \lambda \geq (\overline{x}_1)^{-1} - \lambda > 0$, and $\frac{d}{dx_1}\left(\frac{G(x_1)}{g(x_1)}\right) = 1 > 0$. The optimal payment plan $s_1(x_1)$ can be computed accordingly, which leads to the following results.

Proposition A1 *Suppose the demand distribution $F(\xi)$ is exponential with rate λ and the initial-inventory distribution $G(x_1)$ is uniform over $[0, y_0]$. Then, the optimal order plan is given by:*

$$q_1(x_1) = \begin{cases} \lambda^{-1}\ln\left[\frac{r}{c}\,(1-\lambda x_1)\right] - x_1, & x_1 \in [0, \overline{x}_1 \wedge y_0], \\ 0, & x_1 \in (\overline{x}_1 \wedge y_0, y_0], \end{cases}$$

where $\overline{x}_1 < \lambda^{-1}$ *solves* $\lambda^{-1}\ln\left[\frac{r}{c}(1-\lambda\overline{x}_1)\right] - \overline{x}_1 = 0$. *The optimal payment plan is given by:*

$$s_1(x_1) = \begin{cases} \lambda^{-1}c\left(\frac{1}{1-\lambda(\overline{x}_1\wedge y_0)} - \frac{1}{1-\lambda x_1} + \ln\left(\frac{1-\lambda x_1}{1-\lambda(\overline{x}_1\wedge y_0)}\right)\right), & x_1 \in [0, \overline{x}_1 \wedge y_0], \\ 0, & x_1 \in (\overline{x}_1 \wedge y_0, y_0]. \end{cases}$$

Proof of Proposition A1. The payment plan can be computed as

$$\begin{aligned} s_1(x_1) &= v_1(x_1 + q_1(x_1)) - u_1(x_1) \\ &= v_1(x_1 + q_1(x_1)) - \left[u_1(\overline{x}_1 \wedge y_0) - \int_{x_1}^{\overline{x}_1\wedge y_0} v_1'(\widehat{x}_1 + q_1(\widehat{x}_1))d\widehat{x}_1\right] \\ &= v_1(x_1 + q_1(x_1)) - v_1(\overline{x}_1 \wedge y_0) + \int_{x_1}^{\overline{x}_1\wedge y_0} v_1'(\widehat{x}_1 + q_1(\widehat{x}_1))d\widehat{x}_1 \\ &= \lambda^{-1}r(1 - e^{-\lambda(x_1+q_1(x_1))}) - \lambda^{-1}r(1 - e^{-\lambda(\overline{x}_1\wedge y_0)}) + \int_{x_1}^{\overline{x}_1\wedge y_0} re^{-\lambda(\widehat{x}_1+q_1(\widehat{x}_1))}d\widehat{x}_1 \\ &= \lambda^{-1}r\left(\frac{c/r}{1-\lambda(\overline{x}_1\wedge y_0)} - \frac{c/r}{1-\lambda x_1}\right) + \int_{x_1}^{\overline{x}_1\wedge y_0} \frac{c}{1-\lambda\widehat{x}_1}d\widehat{x}_1 \\ &= \lambda^{-1}c\left(\frac{1}{1-\lambda(\overline{x}_1\wedge y_0)} - \frac{1}{1-\lambda x_1}\right) - \lambda^{-1}c\ln(1-\lambda\widehat{x}_1)\,\Big|_{x_1}^{\overline{x}_1\wedge y_0} \\ &= \lambda^{-1}c\left(\frac{1}{1-\lambda(\overline{x}_1\wedge y_0)} - \frac{1}{1-\lambda x_1} + \ln\left(\frac{1-\lambda x_1}{1-\lambda(\overline{x}_1\wedge y_0)}\right)\right). \end{aligned}$$

■

Second, assume the initial inventory $x_1 = (y_0 - D_0)^+$, where D_0 also follows the exponential distribution with rate λ. In this case, $G(x_1) = e^{-\lambda(y_0-x_1)}$, $0 \le x_1 \le y_0$, with a point mass at 0, and $g(x_1) = \lambda e^{-\lambda(y_0-x_1)}$, $0 < x_1 \le y_0$. The first-order derivative (9) reduces to $\frac{\partial J_1(x_1,q_1)}{\partial q_1} = -c < 0$. Theorem 1 then implies that $q_1(x_1) = 0$ for $x_1 \in (0, y_0]$ and $q_1(0)$ solves $1 - e^{-\lambda q_1} = \frac{r-c}{r}$. The optimal payment plan can be derived accordingly. Thus, we obtain the following results.

Proposition A2 *Suppose the demand distribution* $F(\xi)$ *is exponential with rate* λ *and the initial inventory is given by* $x_1 = (y_0 - D_0)^+$, *where* D_0 *is also exponentially distributed with rate* λ. *Then, the optimal order and payment plans are*

$$q_1(x) = \begin{cases} \lambda^{-1}\ln(\frac{r}{c}), & x_1 = 0, \\ 0, & x_1 \in (0, y_0], \end{cases} \qquad s_1(x_1) = \begin{cases} \lambda^{-1}(r-c), & x_1 = 0, \\ 0, & x_1 \in (0, y_0]. \end{cases}$$

Uniform demand distribution

We next discuss the case in which the demand is uniformly distributed over $[0, \overline{D}]$ (i.e., $F(\xi) = \frac{\xi}{\overline{D}}$ and p.d.f. $f(\xi) = \frac{1}{\overline{D}}$, $\xi \in [0, \overline{D}]$). The retailer's expected revenue and its derivatives are given by:

$$v_1(y_1) = \begin{cases} rE\min\{y_1, \xi\} = ry_1 - \frac{r}{2\overline{D}}y_1^2, & \text{if } y_1 \le \overline{D} \\ \frac{1}{2}r\overline{D}, & \text{if } y_1 \ge \overline{D} \end{cases},$$

$$v_1'(y_1) = \begin{cases} r\overline{F}(y_1) = r(1 - \frac{y_1}{\overline{D}}), & \text{if } y_1 \le \overline{D} \\ 0, & \text{if } y_1 \ge \overline{D} \end{cases}, \quad v_1''(y_1) = \begin{cases} -rf(y_1) = -\frac{r}{\overline{D}}, & \text{if } y_1 \le \overline{D} \\ 0, & \text{if } y_1 \ge \overline{D} \end{cases}.$$

Again, we consider two cases of the initial-inventory distribution $G(x_1)$. First, assume $G(x_1)$ is uniform over $[0, y_0]$. The FOC (8) reduces to:

$$\frac{x_1 + q_1}{\overline{D}} + \frac{x_1}{\overline{D}} = \frac{r-c}{r} \quad \text{or} \quad 2x_1 + q_1 = y_1^*,$$

where $y_1^* = \frac{r-c}{r}\overline{D}$. Let $\overline{x}_1$ be the solution of $2\overline{x}_1 + 0 = y_1^*$, or $\overline{x}_1 = \frac{y_1^*}{2}$ ($\overline{x}_1$ is the smallest x_1 such that $q_1(x_1) = 0$). The optimal quantity plan is therefore given by:

$$q_1(x_1) = \left\{ \begin{array}{cc} y_1^* - 2x_1, & x_1 \in [0, \overline{x}_1] \\ 0, & x_1 \in [\overline{x}_1, y_0] \end{array} \right\}. \tag{A4}$$

We see that $q_1(x_1)$ hits zero at $\frac{y_1^*}{2}$ and $q_1'(x_1) = -2$ on $x_1 \in [0, \frac{y_1^*}{2}]$. The retailer's profit $u_1(x_1)$ can be computed from (4) and the payment plan can be obtained from $s_1(x_1) = v_1(x_1+q_1(x_1)) - u_1(x_1)$.

Next, assume the initial inventory $x_1 = (y_0 - D_0)^+$, where D_0 also follows the distribution $F(\cdot)$. The optimal order quantity $q_1(x_1)$ can be computed from Theorem 1. After some straightforward algebra, we obtain

$$q_1(x_1) = \left\{ \begin{array}{ll} y_1^*, & x_1 = 0 \\ 2\overline{x}_1 - 2x_1, & x_1 \in (0, \overline{x}_1^+] \\ 0, & x_1 \in (\overline{x}_1^+, y_0] \end{array} \right\},$$

where $\overline{x}_1 = \frac{1}{2}(y_0 - \frac{c}{r}\overline{D})$ and $\overline{x}_1^+ = \max\{\overline{x}_1, 0\}$. Notice that $2\overline{x}_1 < y_0 \le \overline{x}_1$, hence $q_1(0^+) < q_1(0)$, which confirms that $q_1(x_1)$ is discontinuous at 0. The retailer's profit $u_1(x_1)$ can be computed from (4) and the payment plan can be obtained from $s_1(x_1) = v_1(x_1 + q_1(x_1)) - u_1(x_1)$. Note that although $u_1(x_1)$ is continuous at 0, $s_1(x_1)$ is not.

To simplify expressions, we make the following assumption with no loss of generality:

Assumption (Normalization). Assume the retail price $r = 1$, the production cost $c \in [0, 1]$, and the inventory holding cost $h \in [0, 1]$. The demand is uniformly distributed on $[0, 1]$, i.e., $\overline{D} = 1$.

Under this assumption, given "previous" inventory position y_0, the distribution of the initial inventory x_1 in period 1 follows $G(x_1|y_0) = x_1 + 1 - y_0,\ x_1 \in [(y_0 - 1)^+, y_0]$. The retailer's expected revenue and its derivatives in period 1 are simplified to:

$$v_1(y_1) = \begin{cases} y_1 - \frac{1}{2}y_1^2, & \text{if } y_1 \le 1 \\ \frac{1}{2}, & \text{if } y_1 \ge 1 \end{cases}, \quad v_1'(y_1) = \begin{cases} 1 - y_1, & \text{if } y_1 \le 1 \\ 0, & \text{if } y_1 \ge 1 \end{cases}, \quad v_1''(y_1) = \begin{cases} -1, & \text{if } y_1 \le 1 \\ 0, & \text{if } y_1 \ge 1 \end{cases}.$$

The period-1 channel-optimal inventory position is $y_1^* = 1 - c$, and the upper bound for positive order is $\overline{x}_1(y_0) = \frac{1}{2}(y_0 - c)$. The solution to the single-period problem can be summarized as below.

Proposition A3 *Suppose previous inventory position is y_0. Under the normalization assumption, the supplier's optimal contract in period 1 can take one of four different forms, depending on y_0:*
(a) $0 \le y_0 \le c$,

$$q_1(x_1|y_0) = \begin{cases} 1-c, & x_1 = 0 \\ 0, & x_1 \in (0, y_0] \end{cases} ; \quad u_1(x_1|y_0) = v_1(x_1),\ x_1 \in [0, y_0] \ ;$$

(b) $c \le y_0 \le 1$,

$$q_1(x_1|y_0) = \begin{cases} 1-c, & x_1 = 0 \\ y_0 - c - 2x_1, & x_1 \in (0, \overline{x}_1(y_0)) \\ 0, & x_1 \in [\overline{x}_1(y_0), y_0] \end{cases} ;$$

$$u_1(x_1|y_0) = \begin{cases} v_1(x_1) + (\overline{x}_1(y_0) - x_1)^2, & x_1 \in [0, \overline{x}_1(y_0)) \\ v_1(x_1), & x_1 \in [\overline{x}_1(y_0), y_0] \end{cases} ;$$

(c) $1 \le y_0 \le 2 - c$,

$$q_1(x_1|y_0) = \begin{cases} y_0 - c - 2x_1, & x_1 \in [y_0 - 1, \overline{x}_1(y_0)) \\ 0, & x_1 \in [\overline{x}_1(y_0), y_0] \end{cases} ;$$

$$u_1(x_1|y_0) = \begin{cases} v_1(x_1) + (\overline{x}_1(y_0) - x_1)^2, & x_1 \in [y_0 - 1, \overline{x}_1(y_0)) \\ v_1(x_1), & x_1 \in [\overline{x}_1(y_0), y_0] \end{cases} ;$$

(d) $y_0 \ge 2 - c$,

$$q_1(x_1|y_0) = 0, \quad x_1 \in [y_0 - 1, y_0]; \qquad u_1(x_1|y_0) = v_1(x_1), \quad x_1 \in [y_0 - 1, y_0].$$

The optimal contract is illustrated in Figure A1.

Proof of Proposition A3. The order quantity plan $q_1(x_1|y_0)$ is derived in the previous section. The retailer's profit function $u_1(x_1|y_0)$ follows from (4): when $x_1 \in [\overline{x}_1(y_0)^+, y_0]$, $u_1(x_1 \mid y_0) = v_1(x_1)$; when $x_1 \in [(y_0 - 1)^+, \overline{x}_1(y_0)^+)$ and $\overline{x}_1(y_0) > 0$ (otherwise the interval $[(y_0 - 1)^+, \overline{x}_1(y_0)^+)$

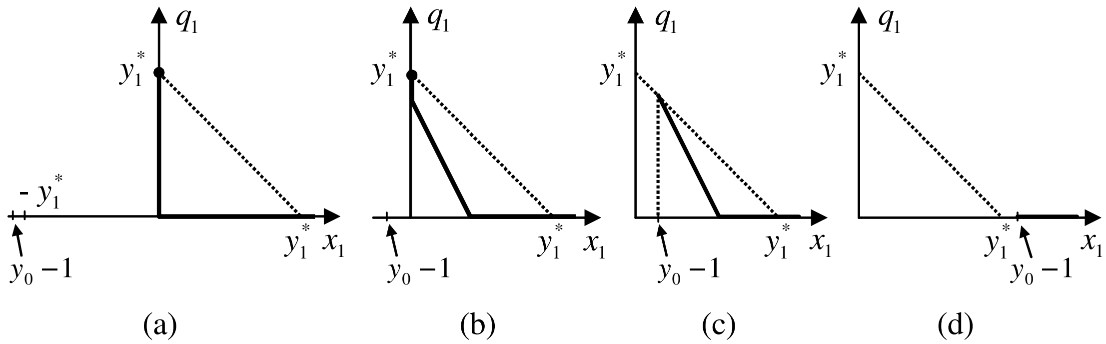

Figure A1: Optimal quantity plan $q_1(x_1|y_0)$ under uniform demand: (a) $0 \le y_0 \le c$; (b) $c \le y_0 \le 1$; (c) $1 \le y_0 \le 2-c$; (d) $y_0 \ge 2-c$.

will be empty),

$$
\begin{aligned}
u_1(x_1 \mid y_0) &= v_1(\overline{x}_1(y_0)) - \int_{x_1}^{\overline{x}_1(y_0)} v_1'\left(z_1 + q_1(z_1|y_0)\right) dz_1 \\
&= v_1(\overline{x}_1(y_0)) - \int_{x_1}^{\overline{x}_1(y_0)} [1 - z_1 - q_1(z_1|y_0)] dz_1 \\
&= v_1(\overline{x}_1(y_0)) - \int_{x_1}^{\overline{x}_1(y_0)} (1 - z_1) dz_1 + \int_{x_1}^{\overline{x}_1(y_0)} q_1(z_1|y_0) dz_1 \\
&= v_1(x_1) + \int_{x_1}^{\overline{x}_1(y_0)} q_1(z_1|y_0) dz_1 \\
&= v_1(x_1) + 2\int_{x_1}^{\overline{x}_1(y_0)} (\overline{x}_1(y_0) - z_1) dz_1 \\
&= v_1(x_1) + (\overline{x}_1(y_0) - x_1)^2.
\end{aligned}
$$

■

Normal demand distribution

The analysis for this case is omitted due to space limitations. It can be obtained from the authors upon request.

Appendix B - Proofs for the Multi-Period Model

Proof of Proposition 1. Let $g_t(x_t)$ be the p.d.f. corresponding to the c.d.f. $G_t(x_t)$, which exists on $(0, \overline{x}_t)$. The objective function (10a) can be rewritten as

$$\begin{aligned}
&\int_0^{\overline{x}_t} \{s_t(x_t) - cq_t(x_t) + \delta\Pi_{t+1}(x_t + q_t(x_t))\} dG_t(x_t) \\
= &\int_0^{\overline{x}_t} \{v_t(x_t + q_t(x_t)) - cq_t(x_t) + \delta\Psi_{t+1}(x_t + q_t(x_t)) - u_t(x_t \mid G_t)\} dG_t(x_t) \\
= &\int_0^{\overline{x}_t} \left\{ v_t(x_t + q_t(x_t)) - cq_t(x_t) + \delta\Psi_{t+1}(x_t + q_t(x_t)) + \int_{x_t}^{\overline{x}_t} u_t'(z \mid G_t) dz \right\} dG_t(x_t) \\
&- u_t(\overline{x}_t \mid G_t) G_t(\overline{x}_t) \\
= &\int_0^{\overline{x}_t} \{v_t(x_t + q_t(x_t)) - cq_t(x_t) + \delta\Psi_{t+1}(x_t + q_t(x_t))\} dG_t(x_t) + \int_0^{\overline{x}_t} u_t'(z \mid G_t) G_t(z) dz \\
&- u_t(\overline{x}_t \mid G_t) G_t(\overline{x}_t) \\
= &\int_{0^+}^{\overline{x}_t} \left\{ v_t(x_t + q_t(x_t)) - cq_t(x_t) + \delta\Psi_{t+1}(x_t + q_t(x_t)) + u_t'(x_t \mid G_t) \frac{G_t(x_t)}{g_t(x_t)} \right\} g_t(x_t) dx_t \\
&+ \{v_t(q_t(0)) - cq_t(0) + \delta\Psi_{t+1}(q_t(0))\} G_t(0) - u_t(\overline{x}_t \mid G_t) G_t(\overline{x}_t) \\
= &\int_{0^+}^{\overline{x}_t} \left\{ \begin{array}{c} v_t(y_t(x_t)) - cy_t(x_t) + \delta\Psi_{t+1}(y_t(x_t)) \\ + \left(v_t'(y_t(x_t)) + \delta \frac{\partial U_{t+1}(y_t(x_t) \mid y_t(x_t))}{\partial y_t} \right) \frac{G_t(x_t)}{g_t(x_t)} \end{array} \right\} g(x_t) dx_t + \int_{0^+}^{\overline{x}_t} cx_t g_t(x_t) dx_t \\
&+ \{v_t(q_t(0)) - cq_t(0) + \delta\Psi_{t+1}(q_t(0))\} G_t(0) - u_t(\overline{x}_t \mid G_t) G_t(\overline{x}_t).
\end{aligned}$$

The above analysis used two facts: (1) the distribution $G_t(x_t)$ may have a point mass at $x_t = 0$ and thus $dG_t(0) = G_t(0) > 0$ in general; (2) $\int_0^{\overline{x}_t} \int_{x_t}^{\overline{x}_t} u_t'(z \mid G_t) dz dG_t(x_t) = \int_0^{\overline{x}_t} u_t'(z \mid G_t) G_t(z) dz$, by switching the order of integrations.

The above expression implies that in the separating region the optimal order plan, $q_t(x_t)$, or the optimal post-order inventory level, $y_t(x_t)$, maximizes $J_t(y_t \mid x_t)$ for any $x_t \in [0, \overline{x}_t]$ subject to $y_t \geq x_t$. The FOC and the lower-bound constraint imply the results. The first-order derivative $\frac{\partial J(y_t \mid x_t)}{\partial y_t}$, which is used later in our analysis, is given by

$$\frac{\partial J(y_t \mid x_t)}{\partial y_t} = \left\{ \begin{array}{lr} v_t'(y_t) - c + \delta\Psi_{t+1}'(y_t) + \left(v_t''(y_t) + \delta \frac{d}{dy_t} \left(\frac{\partial U_{t+1}(y_t \mid y_t)}{\partial y_t} \right) \right) \frac{G_t(x_t)}{g_t(x_t)}, & x_t \in (0, \overline{x}_t] \\ v_t'(y_t) - c + \delta\Psi_{t+1}'(y_t), & x_t = 0 \end{array} \right. . \tag{B1}$$

■

Proof of Theorem 3. Recall that, for any fixed y_{t-1}, the distribution $G_t(x_t|y_{t-1}) = e^{-\lambda(y_{t-1} - x_t)}$, for $0 \leq x_t \leq y_{t-1}$, and $g_t(x_t|y_{t-1}) = \lambda e^{-\lambda(y_{t-1} - x_t)}$, for $0 < x_t \leq y_{t-1}$, which satisfies $\frac{G_t(x_t|y_{t-1})}{g_t(x_t|y_{t-1})} =$

λ^{-1}, for $0 < x_t \le y_{t-1}$. We generalize the domains of G_t and g_t such that $G_t(x_t|y_{t-1}) = 1$ and $g_t(x_t|y_{t-1}) = 0$ for $0 \le y_{t-1} < x_t$. Then, we have $g_t(x_t|y_{t-1}) = \frac{\partial G_t(x_t|y_{t-1})}{\partial x_t}$ for all $x_t > 0$ and $y_{t-1} \ge 0$ and

$$\frac{G_t(x_t|y_{t-1})}{g_t(x_t|y_{t-1})} = \left\{ \begin{array}{ll} \lambda^{-1}, & 0 < x_t \le y_{t-1} \\ +\infty, & 0 \le y_{t-1} < x_t \end{array} \right\} \ge \lambda^{-1}, \text{ for } x_t > 0 \text{ and } y_{t-1} \ge 0. \tag{B2}$$

Because

$$G_t(x_t|y_{t-1} \in q_{t-1} + S_{t-1}) = \frac{\int_{x_{t-1}\in S_{t-1}} G_t(x_t|x_{t-1} + q_{t-1}) dG_{t-1}(x_{t-1})}{\int_{x_{t-1}\in S_{t-1}} dG_{t-1}(x_{t-1})},$$
$$g_t(x_t|y_{t-1} \in q_{t-1} + S_{t-1}) = \frac{\int_{x_{t-1}\in S_{t-1}} g_t(x_t|x_{t-1} + q_{t-1}) dG_{t-1}(x_{t-1})}{\int_{x_{t-1}\in S_{t-1}} dG_{t-1}(x_{t-1})},$$

expression (B2) implies that

$$\frac{G_t(x_t|y_{t-1} \in q_{t-1} + S_{t-1})}{g_t(x_t|y_{t-1} \in q_{t-1} + S_{t-1})} = \frac{\int_{x_{t-1}\in S_{t-1}} G_t(x_t|x_{t-1} + q_{t-1}) dG_{t-1}(x_{t-1})}{\int_{x_{t-1}\in S_{t-1}} g_t(x_t|x_{t-1} + q_{t-1}) dG_{t-1}(x_{t-1})} \ge \lambda^{-1}, \text{ for } 0 < x_t \le b, \tag{B3}$$

where $b = \max\{x_{t-1} + q_{t-1} : x_{t-1} \in S_{t-1}\}$. ■

Proof of Proposition 2. Given belief $G_T(x_T)$ of the beginning inventory (assuming $x_T \in [0, \overline{x}_T]$), because $\Psi'_{T+1}(\cdot) = U''_{T+1}(\cdot) = 0$, $v'_T(y_T) = re^{-\lambda y_T}$, and $v''_T(y_T) = -r\lambda e^{-\lambda y_T}$, the first-order partial derivative of the virtual surplus becomes

$$\partial J_T(y_T \mid x_T)/\partial y_T = \begin{cases} re^{-\lambda y_T} - c, & x_T = 0, \\ r\left[1 - \lambda \frac{G_T(x_T)}{g_T(x_T)}\right] e^{-\lambda y_T} - c, & x_T \in (0, \overline{x}_T]. \end{cases} \tag{B4}$$

Thus, $q_T(0)$ satisfies $re^{-\lambda y_T^*} - c = 0$, or, $y_T^* = \lambda^{-1}\ln(\frac{r}{c})$. By Theorem 3, $G_T(x_T)$ is weakly reverse exponential and hence $\frac{G_T(x_T)}{g_T(x_T)} \ge \lambda^{-1}$ for $x_T > 0$. Thus, when $x_T \in (0, \overline{x}_T]$, $\frac{\partial J_T(y_T \mid x_T)}{\partial y_T} \le -c < 0$ for all $y_T \ge 0$, implying that $q_T(x_T) = 0$. The payment function $s_T(\cdot)$ can be determined accordingly. ■

Proof of Lemma 1. By definition and Proposition 2,

$$\begin{aligned}
U_T(y_{T-1}) &= \int_{0^+}^{y_{T-1}} v_T(x_T)\lambda e^{-\lambda(y_{T-1}-x_T)} dx_T = re^{-\lambda y_{T-1}} \int_{0^+}^{y_{T-1}} (e^{\lambda x_T} - 1) dx_T \\
&= re^{-\lambda y_{T-1}}(\lambda^{-1}(e^{\lambda y_{T-1}} - 1) - y_{T-1}) = \lambda^{-1}r - \lambda^{-1}(r + r\lambda y_{T-1})e^{-\lambda y_{T-1}}, \\
\Psi_T(y_{T-1}) &= (v_T(y_T^*) - cy_T^*)e^{-\lambda y_{T-1}} + \int_{0^+}^{y_{T-1}} v_T(x_T)\lambda e^{-\lambda(y_{T-1}-x_T)} dx_T \\
&= \lambda^{-1}(r - c - c\ln(\frac{r}{c}))e^{-\lambda y_{T-1}} + \lambda^{-1}r - \lambda^{-1}(r + r\lambda y_{T-1})e^{-\lambda y_{T-1}} \\
&= \lambda^{-1}r - \lambda^{-1}\left(c + c\ln\left(\frac{r}{c}\right) + r\lambda y_{T-1}\right)e^{-\lambda y_{T-1}}.
\end{aligned}$$

■

Proof of Proposition 3. When $x_{T-1} = 0$, by Lemma 1,

$$\begin{aligned}\frac{\partial J_{T-1}(y_{T-1} \mid x_{T-1})}{\partial y_{T-1}} &= v'_{T-1}(y_{T-1}) - c + \delta \Psi'_T(y_{T-1}) \\ &= (r+h)e^{-\lambda y_{T-1}} - h - c + \delta\left[\lambda r y_{T-1} - r + c + c\ln\left(\frac{r}{c}\right)\right] e^{-\lambda y_{T-1}},\end{aligned}$$

and hence $q_{T-1}(0)$ (or y^*_{T-1}) solves the first-order equation $(c+h)e^{\lambda y} - \delta\lambda r y = (1-\delta)r + \delta c\left(1+\ln\left(\frac{r}{c}\right)\right) + h$. We can verify that the SOC holds.

When $x_{T-1} > 0$, by Lemma 1,

$$\begin{aligned}\frac{\partial J_{T-1}(y_{T-1} \mid x_{T-1})}{\partial y_{T-1}} =& v'_{T-1}(y_{T-1}) - c + \delta\Psi'_T(y_{T-1}) + \lambda^{-1}\left(v''_{T-1}(y_{T-1}) + \delta U''_T(y_{T-1})\right) \\ =&(r+h)e^{-\lambda y_{T-1}} - h - c + \delta\left[\lambda r y_{T-1} - r + c + c\ln\left(\frac{r}{c}\right)\right] e^{-\lambda y_{T-1}} \\ &- (r+h)e^{-\lambda y_{T-1}} + \delta(r - \lambda r y_{T-1})e^{-\lambda y_{T-1}} \\ =&\delta\left(c + c\ln\left(\frac{r}{c}\right)\right) e^{-\lambda y_{T-1}} - c - h.\end{aligned}$$

If $h \geq \delta\left(c + c\ln\left(\frac{r}{c}\right)\right) - c$, $\frac{\partial J_{T-1}(y_{T-1} \mid x_{T-1})}{\partial y_{T-1}} \leq 0$ for all $y_{T-1} \geq 0$ and the optimal order plan is $q_{T-1}(x_{T-1}) = 0$ for $x_{T-1} \in (0, y_{T-2}]$. If $h < \delta(c + c\ln(\frac{r}{c})) - c$, the FOC becomes

$$\delta\left(c + c\ln(\frac{r}{c})\right) e^{-\lambda y} - c - h = 0,$$

and hence the optimal y_{T-1} (the x^*_{T-1} in the proposition) equals $\lambda^{-1}\ln\left[\frac{\delta c}{c+h}\left(1+\ln\left(\frac{r}{c}\right)\right)\right]$, which is valid on $0 < x_{T-1} \leq y_{T-1}$; when $x_{T-1} > \lambda^{-1}\ln\left[\frac{\delta c}{c+h}\left(1+\ln(\frac{r}{c})\right)\right]$, $\frac{\partial J_{T-1}(y_{T-1} \mid x_{T-1})}{\partial y_{T-1}} \leq 0$ for all $y_{T-1} \geq x_{T-1}$ and hence $q_{T-1}(x_{T-1}) = 0$. We can also verify that the SOC holds.

Furthermore, compared with the case $x_{T-1} = 0$, when $x_{T-1} > 0$, the expression of $\frac{\partial J_{T-1}(y_{T-1} \mid x_{T-1})}{\partial y_{T-1}}$ contains an extra part $-(r+h)e^{-\lambda y_{T-1}} + \delta(r - \lambda r y_{T-1})e^{-\lambda y_{T-1}} = -(1-\delta)re^{-\lambda y_{T-1}} - he^{-\lambda y_{T-1}} - \delta\lambda r y_{T-1}e^{-\lambda y_{T-1}} < 0$. This negative term drags the function $\frac{\partial J_{T-1}(y_{T-1} \mid x_{T-1})}{\partial y_{T-1}}$ downward and thus its root must be smaller in this case than in the case of $x_{T-1} = 0$. Therefore, $x^*_{T-1} < y^*_{T-1}$. ■

Proof of Proposition 4. By Theorem 3, the beginning-inventory distribution $G_{T-1}(x_{T-1})$ is weakly reverse exponential and hence $\frac{G_{T-1}(x_{T-1})}{g_{T-1}(x_{T-1})} \geq \lambda^{-1}$. Because $v''_{T-1}(y_{T-1}) + \delta U''_T(y_{T-1}) < 0$, we have $\frac{\partial J_{T-1}(y_{T-1} \mid x_{T-1})}{\partial y_{T-1}} \leq v'_{T-1}(y_{T-1}) - c + \delta\Psi'_T(y_{T-1}) + \lambda^{-1}(v''_{T-1}(y_{T-1}) + \delta U''_T(y_{T-1}))$. Following the proof of Proposition 3, we can show that $\frac{\partial J_{T-1}(y_{T-1} \mid x_{T-1})}{\partial y_{T-1}} \leq 0$ for all $y_{T-1} \geq 0$ when $h \geq \delta(c + c\ln(\frac{r}{c})) - c$ and thus the BOC is optimal. ■

Proof of Proposition 5. The proof is by straightforward algebra.

(1) Consider the optimal contract first. According to Proposition 3, the optimal order quantity is y^*_{T-1} when $x_{T-1} = 0$, is $x^*_{T-1} - x_{T-1}$ when $x_{T-1} \in (0, x^*_{T-1} \wedge y_{T-2}]$, and is zero when $x_{T-1} \in (x^*_{T-1} \wedge y_{T-2}, y_{T-2}]$ (non-empty if $x^*_{T-1} < y_{T-2}$). Thus, the expected channel profit is given by $v_{T-1}(y^*_{T-1}) - cy^*_{T-1} + \delta\Psi_T(y^*_{T-1})$, $v_{T-1}(x^*_{T-1}) - c(x^*_{T-1} - x_{T-1}) + \delta\Psi_T(x^*_{T-1})$, and $v_{T-1}(x_{T-1}) + \delta\Psi_T(x_{T-1})$ in those cases, respectively, resulting in the three parts of $\Psi^{Opt}_{T-1}(y_{T-2})$. The expected retailer-profit function $U^{Opt}_{T-1}(y_{T-2})$ can be derived in three parts as well. When $x_{T-1} \in (0, x^*_{T-1} \wedge y_{T-2}]$, the local incentive-compatibility constraint (14) and the fact $u'_{T-1}(x_{T-1}) = v'_{T-1}(x^*_{T-1}) + \delta U'_T(x^*_{T-1}) = u'_{T-1}(x^*_{T-1})$ (by expression 13) lead to the retailer's expected profit $u_{T-1}(x^*_{T-1} \wedge y_{T-2}) - ((x^*_{T-1} \wedge y_{T-2}) - x_{T-1})u'_{T-1}(x^*_{T-1})$; when $x_{T-1} = 0$, the retailer's profit equals $u_{T-1}(x^*_{T-1} \wedge y_{T-2}) - (x^*_{T-1} \wedge y_{T-2})u'_{T-1}(x^*_{T-1})$, because $u_{T-1}(x_{T-1})$ is continuous at 0; when $x_{T-1} \in (x^*_{T-1} \wedge y_{T-2}, y_{T-2}]$, her expected profit is simply $v_{T-1}(x_{T-1}) + \delta U_T(x_{T-1})$. Thus, the expression of $U^{Opt}_{T-1}(y_{T-2})$ follows.

(2) The profit functions under the BOC are much simpler because the order only occurs at $x_{T-1} = 0$ and the supplier charges the price to make the retailer's expected profit (at $x_{T-1} = 0$) exactly her reservation profit $v_{T-1}(0) + \delta U_T(0)$, which is zero. ■

Proof of Proposition 6. (1) Multiplying both sides of (16) by $e^{\lambda y}$ yields

$$\begin{aligned} e^{\lambda y} U_\infty(y) &= \int_{0^+}^{y} \{v(x) + \delta U_\infty(x)\} \lambda e^{\lambda x} dx + \{v(b) - s + \delta U_\infty(b)\} \\ &= \int_{0^+}^{y} \left\{\lambda e^{\lambda x} v(x) + \delta\lambda e^{\lambda x} U_\infty(x)\right\} dx + \left\{v(b) - s + \delta e^{-\lambda b} e^{\lambda b} U_\infty(b)\right\}. \end{aligned}$$

By a transformation $\widetilde{U}_\infty(y) = e^{\lambda y} U_\infty(y)$, we obtain

$$\widetilde{U}_\infty(y) = \int_{0^+}^{y} \left\{\lambda e^{\lambda x} v(x) + \delta\lambda \widetilde{U}_\infty(x)\right\} dx + \left\{v(b) - s + \delta e^{-\lambda b} \widetilde{U}_\infty(b)\right\}. \tag{B5}$$

This gives rise to a first-order ordinary differential equation (ODE):

$$\begin{aligned} \widetilde{U}'_\infty(y) &= \lambda e^{\lambda y} v(y) + \delta\lambda \widetilde{U}_\infty(y), \\ \widetilde{U}'_\infty(y) - \delta\lambda \widetilde{U}_\infty(y) &= e^{\lambda y}\{(r+h)(1 - e^{-\lambda y}) - \lambda h y\} = -\lambda h y e^{\lambda y} + (r+h)e^{\lambda y} - (r+h). \end{aligned}$$

Through some straightforward algebra, the general solution to this ODE can be found as

$$\widetilde{U}_\infty(y) = -\frac{h}{1-\delta} y e^{\lambda y} + \frac{(1-\delta)r + (2-\delta)h}{(1-\delta)^2 \lambda} e^{\lambda y} + \frac{r+h}{\delta\lambda} + M_u e^{\delta\lambda y}, \tag{B6}$$

in which M_u is a constant to be determined from a boundary condition. Equation (B5) gives the following boundary condition:

$$\widetilde{U}_\infty(0) = \lambda^{-1}(r+h)(1-e^{-\lambda b}) - hb - s + \delta e^{-\lambda b}\widetilde{U}_\infty(b).$$

Substituting the general solution (B6) to this boundary condition, we have:

$$\begin{aligned}
&\frac{(1-\delta)r + (2-\delta)h}{(1-\delta)^2\lambda} + \frac{r+h}{\delta\lambda} + M_u \\
&\quad = \lambda^{-1}(r+h)(1-e^{-\lambda b}) - hb - s + \delta e^{-\lambda b}\left\{-\frac{h}{1-\delta}be^{\lambda b} + \frac{(1-\delta)r+(2-\delta)h}{(1-\delta)^2\lambda}e^{\lambda b} + \frac{r+h}{\delta\lambda} + M_u e^{\delta\lambda b}\right\} \\
&\quad = \frac{r+h}{\lambda} - \frac{h}{1-\delta}b - s + \delta\frac{(1-\delta)r+(2-\delta)h}{(1-\delta)^2\lambda} + \delta M_u e^{-\lambda(1-\delta)b}.
\end{aligned}$$

This, together with $U_\infty(y) = e^{-\lambda y}\widetilde{U}_\infty(y)$, gives (18).

(2) Next, with transformation $\underline{\widetilde{U}}_\infty(y) = e^{\lambda y}\underline{U}_\infty(y)$, we arrive at

$$\underline{\widetilde{U}}_\infty(y) = \int_{0^+}^{y}\left\{\lambda e^{\lambda x}v(x) + \delta\lambda\underline{\widetilde{U}}_\infty(x)\right\}dx.$$

This equation leads to the same ODE as equation (B5) does and hence has the same general solution (B6), but with a different constant $\underline{M}_u$. The constant can be determined from the boundary condition $\underline{\widetilde{U}}_\infty(0) = 0$:

$$\underline{M}_u = -\frac{(1-\delta)r+(2-\delta)h}{(1-\delta)^2\lambda} - \frac{r+h}{\delta\lambda} = -\frac{(1-\delta)r+h}{\delta(1-\delta)^2\lambda}.$$

The function $\underline{U}_\infty(\cdot)$ is the same as $U_\infty(\cdot)$ in expression (18) after the constant is replaced by $\underline{M}_u$.

(3) By a similar transformation, $\widetilde{\Pi}_\infty(y) = e^{\lambda y}\Pi_\infty(y)$, we obtain from (17) that

$$\widetilde{\Pi}_\infty(y) = \int_{0^+}^{y}\delta\lambda\widetilde{\Pi}_\infty(x)dx + \left\{s - cb + \delta e^{-\lambda b}\widetilde{\Pi}_\infty(b)\right\}.$$

It leads to a simple first-order ODE, $\widetilde{\Pi}'_\infty(y) = \delta\lambda\widetilde{\Pi}_\infty(y)$, which has a general solution $\widetilde{\Pi}_\infty(y) = M_\pi e^{\lambda\delta y}$. The constant M_π can be determined from the boundary condition $\widetilde{\Pi}_\infty(0) = s - cb + \delta e^{-\lambda b}\widetilde{\Pi}_\infty(b)$ as $M_\pi = s - cb + \delta M_\pi e^{-\lambda(1-\delta)b}$. Thus, we obtain (20). The inequality holds because it must be true that $s > cb$ for the supplier to make any profit. Clearly, $\Pi_\infty(y) = e^{-\lambda y}\widetilde{\Pi}_\infty(y) = M_\pi e^{-\lambda(1-\delta)y}$.

(4) Lastly, (21) follow immediately from $\Psi_\infty(y) = U_\infty(y) + \Pi_\infty(y)$ and $M_\psi = M_u + M_\pi$. ■

Proof of Theorem 4. The retailer's information rent is simply given by $U_\infty(y) - \underline{U}_\infty(y) = (M_u - \underline{M}_u)e^{-\lambda(1-\delta)y}$, which is nonnegative if and only if $M_u \geq \underline{M}_u$.

From the supplier's profit-to-go function and (20), we see that the supplier's profit is increasing in s. On the other hand, from expression (18), the constant M_u is decreasing in s. Since M_u is bounded by $\underline{M}_u$ from below, the optimal s^* must force $M_u = \underline{M}_u$, and hence equation (23b) holds, which also implies $U_\infty(y) = \underline{U}_\infty(y)$ for all $y \geq 0$.

Also from (20), the supplier should choose the batch size b to maximize the constant M_π. Since $M_\pi = M_\psi - M_u = M_\psi - \underline{M}_u$ and $\underline{M}_u$ is independent of b, it is equivalent to maximizing M_ψ. From expression (21) and the FOC $dM_\psi/db = 0$, we have

$$\begin{aligned}
\left(\frac{h}{1-\delta} + c\right)\left(1 - \delta e^{-\lambda(1-\delta)b}\right) &= \left(\frac{r}{\delta\lambda} + \frac{h}{\delta(1-\delta)\lambda} + \frac{h}{1-\delta}b + cb\right)\lambda(1-\delta)\delta e^{-\lambda(1-\delta)b}, \qquad \text{(B7)}\\
\frac{h}{1-\delta} + c &= \left(\frac{h}{1-\delta} + c + \lambda(1-\delta)\left(\frac{r}{\delta\lambda} + \frac{h}{\delta(1-\delta)\lambda} + \frac{h}{1-\delta}b + cb\right)\right)\delta e^{-\lambda(1-\delta)b},\\
\frac{h}{1-\delta} + c &= \left(\frac{h}{1-\delta} + c + \frac{1-\delta}{\delta}r + \frac{h}{\delta} + \lambda hb + \lambda(1-\delta)cb\right)\delta e^{-\lambda(1-\delta)b},\\
[h + (1-\delta)c]\frac{e^{\lambda(1-\delta)b}}{\delta(1-\delta)} &= c + \frac{1-\delta}{\delta}r + \frac{h}{\delta(1-\delta)} + \lambda b[h + (1-\delta)c],\\
e^{\lambda(1-\delta)b} - \delta(1-\delta)\lambda b &= 1 + \frac{(1-\delta)^2(r-c)}{h + (1-\delta)c}.
\end{aligned}$$

Since the left-hand side of (23a) is increasing in b (and equals 1 when $b = 0$) and the right hand side is greater than 1, the equation has a unique solution $b^* > 0$. It is straightforward to verify that the SOC $dM_\psi^2/(db)^2 < 0$ holds at b^*. From (23b), s^* is uniquely determined. ∎

Proof of Lemma 2. According to expression (B1), we need to compute

$$\frac{\partial J_t(y_t \mid x_t)}{\partial y_t} = v'(y_t) - c + \delta\Psi'_\infty(y_t) + \left(v''(y_t) + \delta U''_\infty(y_t)\right)\frac{G_t(x_t)}{g_t(x_t)},$$

where the partial derivative $\frac{\partial U_{t+1}(y_t \mid \widehat{y}_t)}{\partial y_t}$ becomes $\frac{dU_\infty(y_t)}{dy_t}$ because the continuation contract from period t is independent of $\widehat{y}_t$. The components in the above expression can be summarized as:

$$v'(y) = (r+h)e^{-\lambda y} - h, \qquad v''(y) = -\lambda(r+h)e^{-\lambda y},$$

$$\begin{aligned}
\Psi'_\infty(y) &= -\frac{h}{1-\delta} - \frac{r+h}{\delta}e^{-\lambda y} - \lambda(1-\delta)M_\psi e^{-\lambda(1-\delta)y},\\
U'_\infty(y) &= -\frac{h}{1-\delta} - \frac{r+h}{\delta}e^{-\lambda y} - \lambda(1-\delta)M_u e^{-\lambda(1-\delta)y},\\
U''_\infty(y) &= \frac{\lambda}{\delta}(r+h)e^{-\lambda y} + \lambda^2(1-\delta)^2 M_u e^{-\lambda(1-\delta)y}.
\end{aligned}$$

Because $\frac{G_t(x_t)}{g_t(x_t)} \geq \lambda^{-1}$ from Theorem 3, we obtain:

$$
\begin{aligned}
&\frac{\partial J_t(y_t \mid x_t)}{\partial y_t}\\
=&(r+h)e^{-\lambda y_t} - h - c - \delta\left[\frac{h}{1-\delta} + \frac{r+h}{\delta}e^{-\lambda y_t} + \lambda(1-\delta)M_\psi e^{-\lambda(1-\delta)y_t}\right]\\
&+\left[-\lambda(r+h)e^{-\lambda y_t} + \lambda(r+h)e^{-\lambda y_t} + \lambda^2\delta(1-\delta)^2 M_u e^{-\lambda(1-\delta)y_t}\right]\frac{G_t(x_t)}{g_t(x_t)}\\
\leq&(r+h)e^{-\lambda y_t} - h - c - \delta\left[\frac{h}{1-\delta} + \frac{r+h}{\delta}e^{-\lambda y_t} + \lambda(1-\delta)M_\psi e^{-\lambda(1-\delta)y_t}\right]\\
&+\lambda\delta(1-\delta)^2 M_u e^{-\lambda(1-\delta)y_t}\\
=&-\frac{h}{1-\delta} - c + \lambda\delta(1-\delta)e^{-\lambda(1-\delta)y_t}[(1-\delta)M_u - M_\psi]\\
=&-\frac{h}{1-\delta} - c + \lambda\delta(1-\delta)e^{-\lambda(1-\delta)y_t}[(1-\delta)\underline{M}_u - M_\psi]\\
=&-\frac{h}{1-\delta} - c + \lambda\delta(1-\delta)e^{-\lambda(1-\delta)y_t}\left[\frac{\frac{r}{\delta\lambda} + \frac{h}{\delta(1-\delta)\lambda} + \frac{h}{1-\delta}b + cb}{1-\delta e^{-\lambda(1-\delta)b}} - \frac{(1-\delta)r+h}{\delta(1-\delta)\lambda}\right].
\end{aligned}
\tag{B8}
$$

The inequality above holds because $M_u < 0$. If the term inside the square brackets is non-positive, we immediately have $\frac{\partial J_t(y_t \mid x_t)}{\partial y_t} < 0$ for all y_t; if it is positive, $\frac{\partial J_t(y_t \mid x_t)}{\partial y_t}$ is maximized at $y_t = 0$, so it is sufficient to specify model parameters such that $\frac{\partial J_t(0 \mid x_t)}{\partial y_t} \leq 0$ (i.e., when $y_t = 0$). Note that the requirement $y_t \geq x_t$ in the definition of $J_t(y_t \mid x_t)$ is relaxed here, so the result is stronger than what is really needed.

Equation (B7) leads to

$$
\frac{h+(1-\delta)c}{\lambda(1-\delta)^2\delta}e^{\lambda(1-\delta)b^*} = \frac{\frac{r}{\delta\lambda} + \frac{h}{\delta(1-\delta)\lambda} + \frac{h}{1-\delta}b^* + cb^*}{1-\delta e^{-\lambda(1-\delta)b^*}}.
$$

Substituting $y_t = 0$ into expression (B8) and using the above identity, we obtain

$$
\begin{aligned}
\frac{\partial J_t(0 \mid x_t)}{\partial y_t} &= -\frac{h}{1-\delta} - c + \lambda\delta(1-\delta)\left[\frac{h+(1-\delta)c}{\lambda(1-\delta)^2\delta}e^{\lambda(1-\delta)b^*} - \frac{(1-\delta)r+h}{\delta(1-\delta)\lambda}\right]\\
&= -\frac{h}{1-\delta} - c + \frac{h+(1-\delta)c}{1-\delta}e^{\lambda(1-\delta)b^*} - [(1-\delta)r+h]\\
&= \frac{h+(1-\delta)c}{1-\delta}\left(e^{\lambda(1-\delta)b^*} - 1\right) - [(1-\delta)r+h]\\
&= \frac{h+(1-\delta)c}{1-\delta}\left[e^{\lambda(1-\delta)b^*} - 1 - (1-\delta)\frac{(1-\delta)r+h}{h+(1-\delta)c}\right],
\end{aligned}
$$

and by substituting equation (23a) into the last expression, we have

$$
\frac{\partial J_t(0 \mid x_t)}{\partial y_t} = \frac{h+(1-\delta)c}{1-\delta}\left[\delta(1-\delta)\lambda b^* - \frac{(1-\delta)^2 c + (1-\delta)\,h}{h+(1-\delta)c}\right] = (h+(1-\delta)c)\,(\delta\lambda b^* - 1)\,.
$$

Therefore, the inequality $\frac{\partial J_t(0 \mid x_t)}{\partial y_t} \le 0$ (and hence $\frac{\partial J_t(y_t \mid x_t)}{\partial y_t} < 0$ for all $y_t \ge x_t > 0$) follows from the simple inequality $\delta\lambda b^* \le 1$. ∎

Proof of Theorem 5. We first prove the following claim:

Claim: *Given the assumptions in the theorem, for any $K \ge 3$, if the optimal BOC (b^*, s^*) is offered from period K onward, the optimal contracts in periods 2 through $K-1$ are also the (b^*, s^*) contract.*

We leave the first period out because the initial-inventory distribution G_1 is special and different from G_t of later periods. The proof of the claim is by induction on K:

(Basic step.) Consider the case $K = 3$. By Lemma 2, we need to determine the boundary values of c and h such that $\delta\lambda b^* = 1$. Substituting $\delta\lambda b^* = 1$ into (23a), we have:

$$e^{(1-\delta)/\delta} - (1-\delta) = 1 + \frac{(1-\delta)^2(r-c)}{h + (1-\delta)c},$$
$$\left[e^{(1-\delta)/\delta} + \delta - 2\right](h/r) + (1-\delta)\left[e^{(1-\delta)/\delta} - 1\right](c/r) = (1-\delta)^2,$$

which gives expression (24). It is easy to verify that any point $(\widetilde{c}, \widetilde{h})$ on or above the specified line will cause $\delta\lambda b^* \le 1$. Therefore, if $(\widetilde{c}, \widetilde{h})$ is in the region bounded by this line and the optimal BOC (b^*, s^*) is offered from period 3 onward, by Lemma 2 it must be the optimal contract in period 2.

(Induction step.) Suppose the claim is true for some $K \ge 3$. Consider the case $K' = K + 1$. Because the number of periods between (and including) periods 3 and $K' - 1$ is the same as that between periods 2 and $K-1$, the induction hypothesis implies that if the (b^*, s^*) contract is offered from period K' onward, the optimal contracts between periods 3 and $K' - 1$ are also the (b^*, s^*) contract. Thus, the (b^*, s^*) contract is now offered from period 3 onward, which brings us back to the basic case of $K = 3$ and the optimal contract in period 2 must also be the (b^*, s^*) contract.

Therefore, the claim is proved. It can be easily extended to the first period because $G_1(0) = 1$ and we only need to verify the optimality of the contract (b^*, s^*) at $x_1 = 0$, which follows directly from Theorem 4. Thus, the theorem is proved. ∎

Appendix C - The Two-Period Model

In this appendix, we study a two-period model in which the retailer's initial inventory in period 1 is zero. Thus, the information asymmetry kicks in only in the second period. The optimal contract in this case can be derived using a backward-induction argument. An important point that will become evident from the analysis is that the supplier is able to extract the entire channel profit in this setting. The retailer's expected profit with two periods-to-go is zero, because the supplier can choose payment s_1 big enough to make the IR constraint binding in the first period. As seen in the single-period model with a point mass at zero, the optimal contract for the second period is not smooth: the supplier would force the retailer to order a large quantity when $x_2 = 0$, but much less if x_2 is positive. When the production and holding costs are sufficiently high, the supplier may not want to sell anything to the retailer if x_2 is positive. (A generalization of this observation motivated our study of the BOCs in the infinite-horizon setting in high-cost domains.)

The system starts in the first period with zero inventory, $x_1 = 0$. Suppose the post-order inventory in the first period is y_1. Assume demand in the first period has c.d.f. $F(\xi)$, $\xi \in [0, \overline{D}]$, with the possibility that $\overline{D} = +\infty$. In a lost-sales system, the beginning inventory in the second period, $x_2 = \max\{y_1 - D_1, 0\}$, has distribution $G(x_2|y_1) = \overline{F}(y_1 - x_2)$, $x_2 \in [(y_1 - \overline{D})^+, y_1]$, which contains a point mass $\overline{F}(y_1)$ at $x_2 = 0$ when $y_1 < \overline{D}$. Because $G(\cdot|y_1)$ depends on y_1, the contract offered by the supplier in the second period depends on y_1 as well, and thus can be expressed as $\{s_2(x_2|y_1), q_2(x_2|y_1)\}_{x_2 \in [(y_1 - \overline{D})^+, y_1]}$ for any given y_1. Notice that y_1 is also the quantity ordered in the first period and hence known by the supplier, but x_2 is only known by the retailer.

As in the static case, the retailer's second-period revenue function given post-order inventory y_2, without the order payment, is given by $v_2(y_2) = rE[\min\{y_2, D_2\}] = \int_0^{y_2} r\overline{F}(\xi)d\xi$, and her net-profit function is given by $u_2(x_2|y_1) = v_2(x_2 + q_2(x_2|y_1)) - s_2(x_2|y_1)$. We denote by $\Pi_2(y_1)$ the supplier's expected profit in the second period under the optimal second-period contract given y_1, by $U_2(y_1)$ the corresponding expected profit of the retailer, and by $\Psi_2(y_1) = \Pi_2(y_1) + U_2(y_1)$ the expected channel profit. Given the first period post-order inventory y_1, these profit functions can be written as:

$$U_2(y_1) = \int_{(y_1-\overline{D})^+}^{y_1} u_2(x_2|y_1) dG(x_2|y_1), \tag{C1}$$

$$\Pi_2(y_1) = \int_{(y_1-\overline{D})^+}^{y_1} [v_2(x_2 + q_2(x_2|y_1)) - u_2(x_2|y_1) - cq_2(x_2|y_1)] dG(x_2|y_1), \tag{C2}$$

$$\Psi_2(y_1) = \int_{(y_1-\overline{D})^+}^{y_1} [v_2(x_2 + q_2(x_2|y_1)) - cq_2(x_2|y_1)] dG(x_2|y_1). \tag{C3}$$

In the first period, the retailer's profit is reduced by the cost of the leftover inventory. Let h denote the retailer's unit holding cost. Her revenue in the first period adjusted by the inventory cost is given by $v_1(y_1) = rE\min\{y_1, D_1\} - hE(y_1 - D_1)^+$. We can see that $v_1(y_1)$ is concave and the derivatives $v_1'(y_1)$ and $v_1''(y_1)$ are well-behaved. The supplier chooses the optimal inventory y_1 (or order quantity q_1) and payment s_1 to maximize his expected profit, as the following:

$$\max_{s_1 \in \mathbb{R}, y_1 \geq 0} s_1 - cy_1 + \delta\Pi_2(y_1)$$
$$\text{s.t.} \quad v_1(y_1) - s_1 + \delta U_2(y_1) \geq 0.$$

By choosing s_1 optimally – i.e., letting $s_1 = v_1(y_1) + \delta U_2(y_1)$ – the problem can be simplified to $\max_{y_1 \geq 0}\{v_1(y_1) - cy_1 + \delta\Psi_2(y_1)\}$. That is, the supplier should maximize the channel profit $J_1(y_1) \equiv v_1(y_1) - cy_1 + \delta\Psi_2(y_1)$ in the first period. Note, however, that $J_1(y_1)$ is inferior to the first-best channel profit due to information asymmetry in the second period.

The first-order condition (FOC) along with the constraint $y_1 \geq 0$ imply that the optimal y_1 either equals zero or solves $v_1'(y_1) - c + \delta\Psi_2'(y_1) = 0$, i.e., $r\overline{F}(y_1) - hF(y_1) - c + \delta\Psi_2'(y_1) = 0$. In this equation, the most intriguing term is $\Psi_2'(y_1)$. The expression of $\Psi_2'(y_1)$ can be derived directly under some demand distributions, such as exponential and uniform; under a general demand distribution, however, $\Psi_2'(y_1)$ does not have a simple expression, as can be seen below.

General Case: General Demand Distribution

In this subsection, we express $\Psi_2'(y_1)$ directly from model parameters, under general demand distributions. This is the key step in determining the optimal inventory y_1 in the first period. For comparison purposes, we conduct this analysis for the first-best (FB) and second-best (SB) scenarios simultaneously. By "second best," we mean the case with asymmetric information. Note that the first-order condition $r\overline{F}(y_1) - hF(y_1) - c + \delta\Psi_2'(y_1) = 0$ also applies to the FB scenario if we use the FB channel-profit function $\Psi_2^*(y_1)$ in place of $\Psi_2(y_1)$.

The essential difference between the FB and SB scenarios lies at the second-period order plan $q_2(x_2|y_1)$. In the FB scenario, given the optimal second-period order-up-to level $y_2^* = F^{-1}(\frac{r-c}{r})$, if $y_1 - \overline{D} \le y_2^*$, the optimal second-period order plan is

$$q_2(x_2|y_1) = \begin{cases} y_2^* - x_2, & x_2 \in [y_1 - \overline{D}, y_2^*] \\ 0, & x_2 \in (y_2^*, y_1] \end{cases}.$$

In the SB scenario, according to the static solution, $q_2(x_2|y_1)$ solves the equation

$$F(x_2+q_2) + f(x_2+q_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)} = \frac{r-c}{r}. \tag{C4}$$

Define $\overline{x}_2(y_1)$ as the threshold inventory level so that $q_2(x_2|y_1) = 0$ if and only if $x_2 \ge \overline{x}_2(y_1)$. Then, in the FB scenario when $y_1 - \overline{D} \le y_2^*$, $\overline{x}_2(y_1)$ equals y_2^* and satisfies $F(\overline{x}_2) = \frac{r-c}{r}$; in the SB scenario, by equation (C4) and $G(x_2|y_1) = \overline{F}(y_1 - x_2)$, $\overline{x}_2(y_1)$ satisfies $F(\overline{x}_2) + f(\overline{x}_2)\frac{\overline{F}(y_1-\overline{x}_2)}{f(y_1-\overline{x}_2)} = \frac{r-c}{r}$. We can show the following properties for $q_2(x_2|y_1)$ and $\overline{x}_2(y_1)$:

Lemma C1 *In the FB scenario,*

$$\frac{\partial}{\partial x_2}q_2(x_2|y_1) = -1, \frac{\partial}{\partial y_1}q_2(x_2|y_1) = 0, \text{ and } \frac{d\overline{x}_2(y_1)}{dy_1} = 0.$$

In the SB scenario,

$$\frac{\partial}{\partial x_2}q_2(x_2|y_1) \le -1, \frac{\partial}{\partial y_1}q_2(x_2|y_1) \ge 0 \text{ and } 0 \le \frac{d\overline{x}_2(y_1)}{dy_1} \le 1.$$

Furthermore, in the SB scenario, $q_2(x_2|y_1) = \varepsilon + q_2(x_2+\varepsilon|y_1+\varepsilon)$ *for any* $\varepsilon \in (-x_2, q_2(x_2|y_1)]$.

Proof of Lemma C1. The FB scenario is straightforward; we show the SB scenario in four steps:

(1) Since $q_2(x_2|y_1)$ solves $F(x_2+q_2) + f(x_2+q_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)} = \frac{r-c}{r}$, for any given y_1, the partial derivative with respect to x_2 on the left-hand side gives

$$\begin{aligned}
&f(x_2+q_2)\left(1+\frac{\partial q_2}{\partial x_2}\right) + f'(x_2+q_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\left(1+\frac{\partial q_2}{\partial x_2}\right) + f(x_2+q_2)\frac{\partial}{\partial x_2}\left(\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\right) \\
&= \left(f(x_2+q_2) + f'(x_2+q_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\right)\left(1+\frac{\partial q_2}{\partial x_2}\right) + f(x_2+q_2)\frac{\partial}{\partial x_2}\left(\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\right) \\
&= f(x_2+q_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\left(\frac{f(y_1-x_2)}{\overline{F}(y_1-x_2)} + \frac{f'(x_2+q_2)}{f(x_2+q_2)}\right)\left(1+\frac{\partial q_2}{\partial x_2}\right) + f(x_2+q_2)\frac{\partial}{\partial x_2}\left(\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\right).
\end{aligned}$$

From conditions (6) and (7), we know

$$\frac{f(y_1-x_2)}{\overline{F}(y_1-x_2)} + \frac{f'(x_2+q_2)}{f(x_2+q_2)} \ge 0 \text{ and } \frac{\partial}{\partial x_2}\left(\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\right) \ge 0.$$

Therefore, for the above partial derivative to be 0, we must have $1 + \frac{\partial q_2}{\partial x_2} \le 0$ and thus $\frac{\partial q_2}{\partial x_2} \le -1$.

(2) Given x_2, the partial derivative with respect to y_1 of the left-hand side of $F(x_2+q_2) + f(x_2+q_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)} = \frac{r-c}{r}$ yields

$$\begin{aligned}
& f(x_2+q_2)\frac{\partial q_2}{\partial y_1} + f'(x_2+q_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\frac{\partial q_2}{\partial y_1} + f(x_2+q_2)\frac{\partial}{\partial y_1}\left(\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\right) \\
&= f(x_2+q_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\left(\frac{f(y_1-x_2)}{\overline{F}(y_1-x_2)} + \frac{f'(x_2+q_2)}{f(x_2+q_2)}\right)\frac{\partial q_2}{\partial y_1} - f(x_2+q_2)\frac{\partial}{\partial x_2}\left(\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\right).
\end{aligned}$$

Again, conditions (6) and (7) imply that $\frac{\partial q_2}{\partial y_1} \ge 0$.

(3) Since $\overline{x}_2(y_1)$ satisfies $F(x_2) + f(x_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)} = \frac{r-c}{r}$, the derivative with respect to y_1 on the left-hand side gives

$$\begin{aligned}
& f(x_2)\frac{dx_2}{dy_1} + f'(x_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\frac{dx_2}{dy_1} + f(x_2)\frac{d}{dy_1}\left(\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\right) \\
&= \left(f(x_2) + f'(x_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\right)\frac{dx_2}{dy_1} - f(x_2)\left(\frac{f^2(y_1-x_2) + \overline{F}(y_1-x_2)f'(y_1-x_2)}{f^2(y_1-x_2)}\right)\left(1 - \frac{dx_2}{dy_1}\right) \\
&= f(x_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\left(\frac{f(y_1-x_2)}{\overline{F}(y_1-x_2)} + \frac{f'(x_2)}{f(x_2)}\right)\frac{dx_2}{dy_1} - f(x_2)\frac{\partial}{\partial x_2}\left(\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)}\right)\left(1 - \frac{dx_2}{dy_1}\right).
\end{aligned}$$

Conditions (6) and (7) suggest that the above expression equals zero only if $\frac{dx_2}{dy_1} \ge 0$ and $\frac{dx_2}{dy_1} \le 1$.

(4) Suppose $q_2(x_2|y_1) > 0$ satisfies the FOC (C4): $F(x_2+q_2) + f(x_2+q_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)} = \frac{r-c}{r}$. If y_1 and x_2 are replaced by $y_1+\varepsilon$ and $x_2+\varepsilon$ with a small perturbation ε, the FOC will be satisfied by $q_2(x_2|y_1) - \varepsilon$, because

$$F(x_2+q_2)+f(x_2+q_2)\frac{\overline{F}(y_1-x_2)}{f(y_1-x_2)} = F((x_2+\varepsilon)+(q_2-\varepsilon))+f((x_2+\varepsilon)+(q_2-\varepsilon))\frac{\overline{F}((y_1+\varepsilon)-(x_2+\varepsilon))}{f((y_1+\varepsilon)-(x_2+\varepsilon))}.$$

That is, $q_2(x_2+\varepsilon|y_1+\varepsilon) = q_2(x_2|y_1) - \varepsilon$, for $\varepsilon \in (-x_2, q_2(x_2|y_1))$. The range of ε doesn't include the point $-x_2$ because the above FOC may not be satisfied when the inventory is zero. ■

In order to quantify the impact of y_1 on the second-period order plan $q_2(x_2|y_1)$, we define the function $h(\varepsilon|x_2,y_1) = \varepsilon + q_2(x_2+\varepsilon|y_1+\varepsilon)$. The above lemma leads to the following property of $h(\varepsilon|x_2,y_1)$, which will be used to derive $\Psi_2'(y_1)$ later.

Lemma C2 *In both FB and SB scenarios, we have:*

$$\frac{\partial h(0|x_2,y_1)}{\partial \varepsilon} = 1 + \frac{\partial q_2(x_2|y_1)}{\partial x_2} + \frac{\partial q_2(x_2|y_1)}{\partial y_1} = \begin{cases} 0, & x_2 < \overline{x}_2(y_1) \\ \textit{undefined}, & x_2 = \overline{x}_2(y_1) \\ 1, & x_2 > \overline{x}_2(y_1) \end{cases}.$$

Proof of Lemma C2. (a) Consider the FB scenario. Since the optimal order plan is $q_2(x_2|y_1) = \begin{cases} y_2^* - x_2, & x_2 \in [y_1 - \overline{D}, y_2^*) \\ 0, & x_2 \in [y_2^*, y_1] \end{cases}$, we have $\frac{\partial q_2(x_2|y_1)}{\partial y_1} = 0$ and $\frac{\partial q_2(x_2|y_1)}{\partial x_2} = \begin{cases} -1, & x_2 < y_2^* \\ \text{undefined}, & x_2 = y_2^* \\ 0, & x_2 > y_2^* \end{cases}$. Therefore, the expression for $\frac{\partial h(0|x_2,y_1)}{\partial \varepsilon}$ holds.

(b) Consider the SB scenario, in three cases:

- Suppose $x_2 < \overline{x}_2(y_1)$ and hence $q_2(x_2|y_1) > 0$. From Lemma C1, $q_2(x_2|y_1) = \varepsilon + q_2(x_2 + \varepsilon|y_1 + \varepsilon) = h(\varepsilon|x_2, y_1)$ for $\varepsilon \in [-x_2, q_2(x_2|y_1)]$. As a consequence, $\frac{\partial h(0|x_2,y_1)}{\partial \varepsilon} = 0$.
- If we assume $x_2 > \overline{x}_2(y_1)$, we have $q_2(x_2|y_1) = 0$ and the FOC is violated at $q_2(x_2|y_1)$. When ε is small enough, we must have $q_2(x_2 + \varepsilon|y_1 + \varepsilon) = 0$ as well, and hence $h(\varepsilon|x_2, y_1) = \varepsilon$. Consequently, $\frac{\partial h(0|x_2,y_1)}{\partial \varepsilon} = 1$.
- At the threshold level $x_2 = \overline{x}_2(y_1)$, we have $q_2(x_2|y_1) = 0$ and the FOC is still satisfied. For any $\varepsilon > 0$, the perturbed FOC is satisfied by $q_2(x_2|y_1) - \varepsilon < 0$, which implies $q_2(x_2 + \varepsilon|y_1 + \varepsilon) = 0$ due to the boundary condition and thus $\lim_{\varepsilon \to 0^+} \frac{h(\varepsilon|x_2,y_1) - h(0|x_2,y_1)}{\varepsilon} = 1$; for any $\varepsilon < 0$, we still have $q_2(x_2 + \varepsilon|y_1 + \varepsilon) = q_2(x_2|y_1) - \varepsilon > 0$ and $\lim_{\varepsilon \to 0^-} \frac{h(\varepsilon|x_2,y_1) - h(0|x_2,y_1)}{\varepsilon} = 0$. So, $\frac{\partial h(0|x_2,y_1)}{\partial \varepsilon}$ is undefined at $x_2 = \overline{x}_2(y_1)$. ■

Now, we derive an expression for $\Psi_2'(y_1)$ which contains the term $\partial h(0|x_2, y_1)/\partial \varepsilon$.

Lemma C3 *In both FB and SB scenarios, if the optimal order plan in the second period is denoted by $q_2(x_2|y_1)$ given first-period inventory y_1, the derivative of $\Psi_2(y_1)$ is given by:*

$$\Psi_2'(y_1) = \begin{cases} \int_{0^+}^{y_1} \left[r\overline{F}(x_2 + q_2(x_2|y_1)) - c\right] \frac{\partial h(0|x_2,y_1)}{\partial \varepsilon} f(y_1 - x_2) dx_2 & \\ \quad + cF(y_1) + \left[v_2(q_2(x_2|y_1)) - cq_2(x_2|y_1)\right]\Big|_{x_2=0}^{x_2=0^+} f(y_1), & 0 \le y_1 \le \overline{D} \\ \int_{y_1-\overline{D}}^{y_1} \left[r\overline{F}(x_2 + q_2(x_2|y_1)) - c\right] \frac{\partial h(0|x_2,y_1)}{\partial \varepsilon} f(y_1 - x_2) dx_2 + c, & y_1 \ge \overline{D} \end{cases}.$$

Proof of Lemma C3. The expression (C3) is valid in both FB and SB scenarios. It implies:

$$\Psi_2(y_1) = \begin{cases} \int_{0^+}^{y_1} [v_2(x_2 + q_2(x_2|y_1)) - cq_2(x_2|y_1)] f(y_1 - x_2) dx_2 & \\ \quad + [v_2(q_2(0|y_1)) - cq_2(0|y_1)]\overline{F}(y_1), & 0 \le y_1 \le \overline{D} \\ \int_{y_1-\overline{D}}^{y_1} [v_2(x_2 + q_2(x_2|y_1)) - cq_2(x_2|y_1)] f(y_1 - x_2) dx_2, & y_1 \ge \overline{D} \end{cases}.$$

(1) Consider the case $0 \le y_1 \le \overline{D}$. In this case, the contract may be discontinuous at 0 where we must pay special attention. Specifically, we assume $q_2(0|y_1) \ne q_2(0^+|y_1)$ without loss of generality.

Notice that $q_2(0|y_1) = F^{-1}(\frac{r-c}{r})$, which is independent of y_1. We first show the following fact:

$$\begin{aligned}
&\frac{d}{dy}\left(\int_{0^+}^{y} \mu(x|y)f(y-x)dx + \mu(0|y)\overline{F}(y)\right)\\
&= \int_{0^+}^{y} \frac{\partial\left[\mu(x|y)f(y-x)\right]}{\partial y}dx + \mu(y|y)f(0) + \frac{d}{dy}\left[\mu(0|y)\overline{F}(y)\right]\\
&= \int_{0^+}^{y} \frac{\partial \mu(x|y)}{\partial y}f(y-x)dx + \int_{0^+}^{y} \mu(x|y)f'(y-x)dx + \mu(y|y)f(0) - \mu(0|y)f(y) + \overline{F}(y)\frac{d}{dy}\mu(0|y)\\
&= \int_{0^+}^{y} \frac{\partial \mu(x|y)}{\partial y}f(y-x)dx + \int_{0^+}^{y} \frac{\partial \mu(x|y)}{\partial x}f(y-x)dx + \mu(x|y)|_{0}^{0^+} f(y) + \overline{F}(y)\frac{d}{dy}\mu(0|y)\\
&= \int_{0^+}^{y} \left(\frac{\partial \mu(x|y)}{\partial x} + \frac{\partial \mu(x|y)}{\partial y}\right) f(y-x)dx + \mu(x|y)|_{x=0}^{x=0^+} f(y) + \overline{F}(y)\frac{d}{dy}\mu(0|y).
\end{aligned}$$

This fact implies:

$$\begin{aligned}
\Psi_2'(y_1) &= \int_{0^+}^{y_1-\underline{D}} \left[v_2'(x_2+q_2(x_2|y_1)) - c\right]\left(1 + \frac{\partial q_2(x_2|y_1)}{\partial x_2} + \frac{\partial q_2(x_2|y_1)}{\partial y_1}\right) f(y_1-x_2)dx_2\\
&\quad + \int_{0^+}^{y_1-\underline{D}} cf(y_1-x_2)dx_2 + \left[v_2(x_2+q_2(x_2|y_1)) - cq_2(x_2|y_1)\right]|_{x_2=0}^{x_2=0^+} f(y_1)\\
&= \int_{0^+}^{y_1-\underline{D}} \left[r\overline{F}(x_2+q_2(x_2|y_1)) - c\right] \frac{\partial h(0|x_2,y_1)}{\partial \varepsilon} f(y_1-x_2)dx_2 + cF(y_1)\\
&\quad + \left[v_2(q_2(x_2|y_1)) - cq_2(x_2|y_1)\right]|_{x_2=0}^{x_2=0^+} f(y_1).
\end{aligned}$$

(2) Consider the case $y_1 \geq \overline{D}$, where there is no discontinuity in the contract. We first show the following:

$$\begin{aligned}
&\frac{d}{dy}\left(\int_{y-\overline{D}}^{y} \mu(x|y)f(y-x)dx\right)\\
&= \int_{y-\overline{D}}^{y} \frac{\partial\left[\mu(x|y)f(y-x)\right]}{\partial y}dx + \left[\mu(x|y)f(y-x)\right]|_{x=y-\overline{D}}^{x=y}\\
&= \int_{y-\overline{D}}^{y} \frac{\partial \mu(x|y)}{\partial y}f(y-x)dx + \int_{y-\overline{D}}^{y} \mu(x|y)f'(y-x)dx + \left[\mu(x|y)f(y-x)\right]|_{x=y-\overline{D}}^{x=y}\\
&= \int_{y-\overline{D}}^{y} \frac{\partial \mu(x|y)}{\partial y}f(y-x)dx + \int_{y-\overline{D}}^{y} \frac{\partial \mu(x|y)}{\partial x}f(y-x)dx\\
&= \int_{y-\overline{D}}^{y} \left(\frac{\partial \mu(x|y)}{\partial y} + \frac{\partial \mu(x|y)}{\partial x}\right) f(y-x)dx.
\end{aligned}$$

This result implies:

$$\begin{aligned}
\Psi_2'(y_1) &= \int_{y_1-\overline{D}}^{y_1} \left[v_2'(x_2+q_2(x_2|y_1)) - c\right]\left(1 + \frac{\partial q_2(x_2|y_1)}{\partial x_2} + \frac{\partial q_2(x_2|y_1)}{\partial y_1}\right) f(y_1-x_2)dx_2\\
&\quad + \int_{y_1-\overline{D}}^{y_1} cf(y_1-x_2)dx_2\\
&= \int_{y_1-\overline{D}}^{y_1} \left[r\overline{F}(x_2+q_2(x_2|y_1)) - c\right] \frac{\partial h(0|x_2,y_1)}{\partial \varepsilon} f(y_1-x_2)dx_2 + c.
\end{aligned}$$

It completes the proof of the lemma. ■

According to Lemma C2, the term $\partial h(0|x_2, y_1)/\partial\varepsilon$ is either 0 or 1 almost everywhere. Thus, the expression of $\Psi_2'(y_1)$ in Lemma C3 can be simplified so that it only involves model parameters, as described in the next two corollaries.

Corollary C1 *In the FB scenario, $\Psi_2'(y_1)$ is given by:*

$$\Psi_2'(y_1) = \begin{cases} cF(y_1) - f(y_1)\int_0^{y_2^*}\left[r\overline{F}(\xi) - c\right]d\xi, & 0 \le y_1 \le y_2^*, \\ \int_{y_2^*}^{y_1}\left[r\overline{F}(x_2) - c\right]f(y_1 - x_2)dx_2 + cF(y_1), & y_2^* \le y_1 \le \overline{D}, \\ \int_{y_2^*}^{y_1}\left[r\overline{F}(x_2) - c\right]f(y_1 - x_2)dx_2 + c, & \overline{D} \le y_1 \le y_2^* + \overline{D}, \end{cases}$$

or equivalently,

$$\Psi_2'(y_1) = \begin{cases} cF(y_1) - f(y_1)\int_0^{y_2^*}\left[r\overline{F}(\xi) - c\right]d\xi, & 0 \le y_1 \le y_2^*, \\ cF(y_1) - \int_{y_2^*}^{y_1} rf(x_2)F(y_1 - x_2)dx_2, & y_2^* \le y_1 \le y_2^* + \overline{D}. \end{cases} \tag{C5}$$

By default, $F(y_1) = 1$ when $y_1 > \overline{D}$.

Proof of Corollary C1. The first part of the corollary is obvious. For the second part, when $y_2^* \le y_1 \le y_2^* + \overline{D}$, we have:

$$\begin{aligned} \Psi_2'(y_1) &= \int_{y_2^*}^{y_1}\left[r\overline{F}(x_2) - c\right]f(y_1 - x_2)dx_2 + cF(y_1) \\ &= -\left[r\overline{F}(x_2) - c\right]F(y_1 - x_2)\Big|_{y_2^*}^{y_1} - \int_{y_2^*}^{y_1} rf(x_2)F(y_1 - x_2)dx_2 + cF(y_1) \\ &= cF(y_1) - \int_{y_2^*}^{y_1} rf(x_2)F(y_1 - x_2)dx_2. \end{aligned}$$

■

Corollary C2 *In the SB scenario, $\Psi_2'(y_1)$ is given by:*

$$\Psi_2'(y_1) = \begin{cases} \begin{array}{c}\int_0^{y_1}\left[r\overline{F}(x_2) - c\right]f(y_1 - x_2)dx_2 + cF(y_1) \\ -f(y_1)\int_0^{y_2^*}\left[r\overline{F}(\xi) - c\right]d\xi,\end{array} & 0 \le y_1 \le y_1^0, \\ \begin{array}{c}\int_{\overline{x}_2(y_1)}^{y_1}\left[r\overline{F}(x_2) - c\right]f(y_1 - x_2)dx_2 + cF(y_1) \\ -f(y_1)\int_{q_2(0^+|y_1)}^{q_2(0|y_1)}\left[r\overline{F}(\xi) - c\right]d\xi,\end{array} & y_1^0 \le y_1 \le \overline{D}, \\ \int_{\overline{x}_2(y_1)}^{y_1}\left[r\overline{F}(x_2) - c\right]f(y_1 - x_2)dx_2 + c, & y_1 \ge \overline{D}, \end{cases} \tag{C6}$$

where $q_2(0|y_1) = y_2^*$, $q_2(0^+|y_1)$ solves $F(q_2) + f(q_2)\frac{\overline{F}(y_1)}{f(y_1)} = \frac{r-c}{r}$, $\overline{x}_2(y_1)$ solves $F(\overline{x}_2) + f(\overline{x}_2)\frac{\overline{F}(y_1 - \overline{x}_2)}{f(y_1 - \overline{x}_2)} = \frac{r-c}{r}$, and y_1^0 solves $F(0) + f(0)\frac{\overline{F}(y_1^0)}{f(y_1^0)} = \frac{r-c}{r}$.

The above results, along with the discussion at the beginning of this appendix, can be summarized as follows.

Theorem C1 *The optimal inventory y_1 in the first period solves the FOC*

$$r\overline{F}(y_1) - hF(y_1) - c + \delta\Psi_2'(y_1) = 0, \tag{C7}$$

where $\Psi_2'(y_1)$ can be computed from expression (C5) in the FB scenario or from expression (C6) in the SB scenario.

Describing $\Psi_2'(y_1)$ from model parameters greatly facilitates our ultimate task of finding y_1. However, evidently, the expressions are still non-trivial. Thus, the optimal contract in the first period (the optimal inventory y_1) does not have a closed-form expression and does not possess any obvious structure under a general demand distribution. Nevertheless, if the demand is exponential or uniform, the optimal solution can be determined explicitly, as shown in the next two subsections. The exponential-demand case is easy, because we have analyzed a more general case in §4.4. The uniform-demand case is still involved, as will be seen.

Special Case: Exponential Demand

The two-period model studied in this appendix is in fact a special case of the last-two-period model studied in §4.4. In the latter model, if the beginning inventory of the second-last period x_{T-1} is derived from $(y_{T-2} - D_{T-2})^+$ with $y_{T-2} = 0$, x_{T-1} must be zero with probability 1 and the model reduces to the two-period model considered in this appendix. Thus, Proposition 3 directly implies the following:

Proposition C1 *In a two-period model with exponential demand and zero initial inventory, the optimal order quantity (inventory level) y_1 in the first period solves*

$$(c+h)e^{\lambda y_1} - \delta\lambda r y_1 = (1-\delta)r + \delta c\left(1 + \ln\left(\frac{r}{c}\right)\right) + h.$$

In addition, Lemma 1 can be rewritten as follows.

Proposition C2 *Given first-period inventory y_1, the expected second-period channel profit $\Psi_2(y_1)$ and retailer's profit $U_2(y_1)$ are given by:*

$$\Psi_2(y_1) = \lambda^{-1}r - \lambda^{-1}\left(c + c\ln\left(\frac{r}{c}\right) + r\lambda y_1\right)e^{-\lambda y_1},$$
$$U_2(y_1) = \lambda^{-1}r - \lambda^{-1}\left(r + r\lambda y_1\right)e^{-\lambda y_1}.$$

The retailer's expected total profit for the two periods is zero because the supplier can choose the first-period payment s_1 so as to make the retailer exactly break even:

$$s_1 = v_1(y_1) + \delta U_2(y_1) = \lambda^{-1}(r + \delta r + h)(1 - e^{-\lambda y_1}) - hy_1 - \delta r y_1 e^{-\lambda y_1}.$$

Further, the supplier's expected total profit for the two periods is the same as the channel profit $J_1(y_1)$:

$$\begin{aligned} J_1(y_1) &= v_1(y_1) - cy_1 + \delta\Psi_2(y_1) \\ &= \lambda^{-1}(r + h)(1 - e^{-\lambda y_1}) - (h + c)y_1 + \delta r\lambda^{-1} - \delta\left[ry_1 + \lambda^{-1}c\left(1 + \ln\left(\frac{r}{c}\right)\right)\right]e^{-\lambda y_1}. \end{aligned}$$

Special Case: Uniform Demand

Now, we consider the uniform demand case with c.d.f. $F(\xi) = \frac{\xi}{\overline{D}}$ and p.d.f. $f(\xi) = \frac{1}{\overline{D}}$, $\xi \in [0, \overline{D}]$. Without loss of generality, we normalize the model parameters as in Appendix A:

Assumption (Normalization). Assume the retail price $r = 1$, the production cost $c \in [0, 1]$, and the inventory holding cost $h \in [0, 1]$. The demand is uniformly distributed on $[0, 1]$, i.e., $\overline{D} = 1$.

The optimal contract in the second period is given by Proposition A3. To find the optimal order quantity y_1 in the first period, we need to solve the FOC (C7). From expression (C6) (or Proposition A3), through straightforward algebra, we can obtain that

$$\Psi_2'(y_1) = \begin{cases} -\frac{1}{2}y_1^2 + y_1 - \frac{1}{2}(1-c)^2, & y_1 \in [0, c] \\ -\frac{7}{8}y_1^2 + \left(\frac{3}{2} + \frac{1}{4}c\right)y_1 - \left(\frac{3}{8}c^2 - \frac{1}{2}c + \frac{1}{2}\right), & y_1 \in [c, 1] \\ \frac{1}{8}y_1^2 - \left(\frac{1}{2} + \frac{3}{4}c\right)y_1 - \frac{3}{8}c^2 + \frac{3}{2}c + \frac{1}{2}, & y_1 \in [1, 2-c] \end{cases}.$$

The case $y_1 \geq 2 - c$ is trivial and irrelevant (as will become clear) and is therefore omitted.

The derivative of the two-period channel profit, $J_1'(y_1) = v_1'(y_1) - c + \delta\Psi_2'(y_1)$, can be computed accordingly. Although the parameters c, δ, and h all affect the optimal solution, it can be demonstrated that the production cost c has the biggest impact on the structure of the solution. So, we will focus on the benchmark case of $\delta = 1$ and $h = 0$ in this subsection and only study the impact of c on the optimal solution. In that case, we have $v_1'(y_1) = \begin{cases} 1 - y_1, & \text{if } y_1 \leq 1 \\ 0, & \text{if } y_1 \geq 1 \end{cases}$, and

$$J_1'(y_1) = \begin{cases} -\frac{1}{2}y_1^2 + \frac{1}{2} - \frac{1}{2}c^2 \equiv K_a(y_1), & y_1 \in [0, c] \\ -\frac{7}{8}y_1^2 + \left(\frac{1}{2} + \frac{1}{4}c\right)y_1 - \frac{3}{8}c^2 - \frac{1}{2}c + \frac{1}{2} \equiv K_b(y_1), & y_1 \in [c, 1] \\ \frac{1}{8}y_1^2 - \left(\frac{1}{2} + \frac{3}{4}c\right)y_1 - \frac{3}{8}c^2 + \frac{1}{2}c + \frac{1}{2} \equiv K_c(y_1), & y_1 \in [1, 2-c] \end{cases}.$$

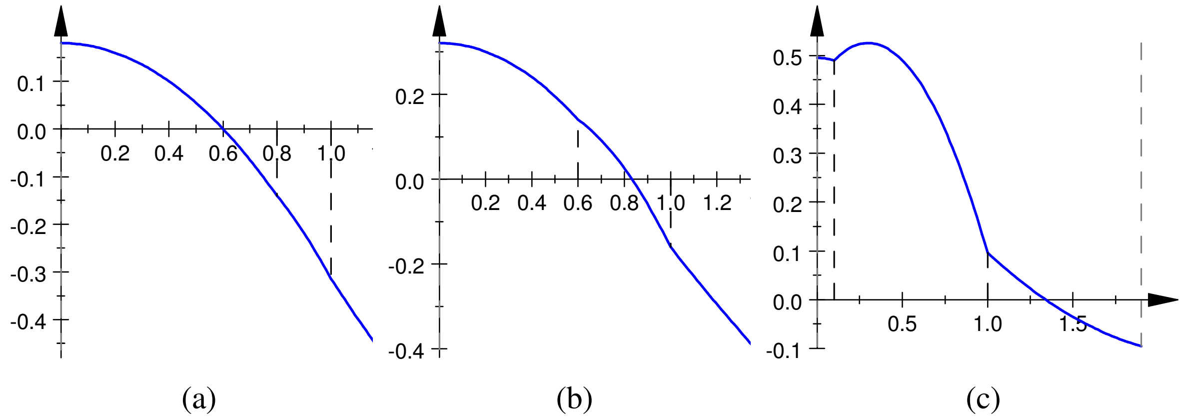

Figure C1: Examples of $J_1'(y_1)$ under uniform demand: (a) $c = 0.8$; (b) $c = 0.6$; (c) $c = 0.1$.

That is, given any $c \in [0,1]$, $J_1'(y_1)$ consists of three segments, $K_a(\cdot)$, $K_b(\cdot)$, and $K_c(\cdot)$, over the domain $[0, 2-c]$. The optimal solution y_1 is the root of function $J_1'(y_1)$, depending on the parameter c. The function $J_1'(y_1)$ is illustrated in Figure C1 for three different values of c, where the root of $J_1'(y_1)$ lies in $[0,c]$, $[c,1]$, and $[1, 2-c]$, respectively.

The optimal y_1 is determined by c as follows.

Proposition C3 *The optimal inventory y_1 in the first period is given by: (a) If $c \in [\frac{\sqrt{2}}{2}, 1)$, $y_1 = \sqrt{1-c^2} \in [0,c]$; (b) If $c \in [\frac{1}{3}, \frac{\sqrt{2}}{2}]$, $y_1 = \frac{1}{7}(2+c) + \frac{2}{7}\sqrt{(2+c)(4-5c)} \in [c,1]$; (c) If $c \in [0, \frac{1}{3}]$, $y_1 = (2+3c) - 2\sqrt{c(2+3c)} \in [1, 2-c]$.*

Proof of Proposition C3. First, we compute the three possible roots y_1^a, y_1^b, and y_1^c, of the three segments $K_a(\cdot)$, $K_b(\cdot)$, and $K_c(\cdot)$, respectively:

(a) From $K_a(y_1) = 0$, we have

$$-\frac{1}{2}y_1^2 + \frac{1}{2} - \frac{1}{2}c^2 = 0,$$

or $y_1^a = \sqrt{1-c^2}$;

(b) From $K_b(y_1) = 0$, we have

$$-\frac{7}{8}y_1^2 + \left(\frac{1}{2} + \frac{1}{4}c\right) y_1 - \frac{3}{8}c^2 - \frac{1}{2}c + \frac{1}{2} = 0,$$
$$y_1^2 - \frac{2}{7}(2+c)y_1 + \frac{1}{7}\left(3c^2 + 4c - 4\right) = 0,$$

or $y_1^b = \frac{1}{7}(2+c) + \frac{2}{7}\sqrt{(2+c)(4-5c)}$;

(c) From $K_c(y_1) = 0$, we have

$$\frac{1}{8}y_1^2 - \left(\frac{1}{2} + \frac{3}{4}c\right) y_1 - \frac{3}{8}c^2 + \frac{1}{2}c + \frac{1}{2} = 0,$$
$$y_1^2 - 2(2+3c)y_1 - 3c^2 + 4c + 4 = 0,$$

or $y_1^c = (2+3c) - 2\sqrt{c(2+3c)}$.

Next, we consider the corresponding range of c in which y_1^a, y_1^b, or y_1^c is the actual root of $J_1'(y_1)$. That happens when y_1^a, y_1^b, or y_1^c lies in the domain of its corresponding segment, $[0, c]$, $[c, 1]$, or $[1, 2-c]$.

(a) When $c \in [\frac{\sqrt{2}}{2}, 1)$, we see that $y_1^a = \sqrt{1-c^2} \in [0, c]$ and hence $y_1 = y_1^a$.

(b) When $c \in [\frac{1}{3}, \frac{\sqrt{2}}{2}]$, we see that $y_1^b \in [c, 1]$, because $c \le \frac{1}{7}(2+c) + \frac{2}{7}\sqrt{(2+c)(4-5c)} \le 1$ is equivalent to $6c - 2 \le 2\sqrt{-5c^2 - 6c + 8} \le 5 - c$, which is exactly $\frac{\sqrt{2}}{2} \ge c \ge \frac{1}{3}$. Thus, $y_1 = y_1^b$.

(c) When $c \in [0, \frac{1}{3}]$, we see that $y_1^c \in [1, 2-c]$, because $1 \le (2+3c) - 2\sqrt{c(2+3c)} \le 2 - c$ is equivalent to $3c + 1 \ge \sqrt{12c^2 + 8c} \ge 4c$, which is exactly $c \le \frac{1}{3}$. Thus, $y_1 = y_1^c$. ■

As discussed earlier, the retailer's expected total profit for the two periods is zero since the supplier can choose the first period payment $s_1 = v_1(y_1) + \delta U_2(y_1)$ to make the IR constraint binding in the first period. For instance, when $h = 0$, s_1 is determined by

$$s_1 = \begin{cases} y_1 - \frac{1}{2}y_1^2 + \delta U_2(y_1), & \text{if } y_1 \le 1 \\ \frac{1}{2} + \delta U_2(y_1), & \text{if } y_1 \ge 1 \end{cases}.$$

The corresponding supplier's expected total profit for the two periods is given by

$$J_1(y_1) = v_1(y_1) - cy_1 + \delta\Psi_2(y_1) = \begin{cases} (1-c)y_1 - \frac{1}{2}y_1^2 + \delta\Psi_2(y_1), & \text{if } y_1 \le 1 \\ \frac{1}{2} - cy_1 + \delta\Psi_2(y_1), & \text{if } y_1 \ge 1 \end{cases}.$$

To complete the analysis, the second-period profit functions $U_2(y_1)$, $\Psi_2(y_1)$ and $\Pi_2(y_1)$ can be computed directly as follows, which concludes this appendix.

Proposition C4 *Given the first period inventory y_1, the expected second-period channel profit $\Psi_2(y_1)$, retailer's profit $U_2(y_1)$, and supplier's profit $\Pi_2(y_1)$ can be computed as follows, in three cases: (a) If $y_1 \in [0, c]$,*

$$U_2(y_1) = \frac{1}{2}y_1^2 - \frac{1}{6}y_1^3,$$
$$\Psi_2(y_1) = \frac{1}{2}y_1^2 - \frac{1}{6}y_1^3 + \frac{1}{2}(1-c)^2(1-y_1),$$
$$\Pi_2(y_1) = \frac{1}{2}(1-c)^2(1-y_1);$$

(b) If $y_1 \in [c,1]$,

$$\begin{aligned}
U_2(y_1) &= \frac{1}{2}y_1^2 - \frac{1}{6}y_1^3 + (1-y_1)\overline{x}_2(y_1)^2 + \frac{1}{3}\overline{x}_2(y_1)^3,\\
\Psi_2(y_1) &= \frac{1}{2}y_1^2 - \frac{1}{6}y_1^3 + (1-y_1)\overline{x}_2(y_1)^2 + \overline{x}_2(y_1)^3 + \frac{1}{2}(1-c)^2(1-y_1),\\
\Pi_2(y_1) &= \frac{2}{3}\overline{x}_2(y_1)^3 + \frac{1}{2}(1-c)^2(1-y_1);
\end{aligned}$$

(c) If $y_1 \in [1, 2-c]$,

$$\begin{aligned}
U_2(y_1) &= \frac{1}{3} + \frac{1}{2}(y_1-1) - \frac{1}{2}(y_1-1)^2 + \frac{1}{6}(y_1-1)^3 + \frac{1}{24}(2-c-y_1)^3,\\
\Psi_2(y_1) &= \frac{1}{3} + \frac{1}{2}(y_1-1) - \frac{1}{2}(y_1-1)^2 + \frac{1}{6}(y_1-1)^3 + \frac{1}{8}(2-c-y_1)^3,\\
\Pi_2(y_1) &= \frac{1}{12}(2-c-y_1)^3.
\end{aligned}$$

Proof of Proposition C4. We compute $U_2(y_1)$, $\Pi_2(y_1)$ and $\Psi_2(y_1)$ following their definitions (C1)-(C3). We consider the three regions of y_1 and apply the result of Proposition A3.

(a) When $y_1 \in [0,c]$, we have

$$\begin{aligned}
U_2(y_1) &= \int_0^{y_1} v_2(x_2)dx_2 + v_2(0)(1-y_1) = \int_0^{y_1}\left(x_2 - \frac{1}{2}x_2^2\right)dx_2 = \frac{1}{2}y_1^2 - \frac{1}{6}y_1^3,\\
\Pi_2(y_1) &= [v_2(1-c) - c(1-c)](1-y_1) = \frac{1}{2}(1-c)^2(1-y_1), \text{ and}\\
\Psi_2(y_1) &= U_2(y_1) + \Pi_2(y_1) = \frac{1}{2}y_1^2 - \frac{1}{6}y_1^3 + \frac{1}{2}(1-c)^2(1-y_1).
\end{aligned}$$

(b) When $y_1 \in [c,1]$, noticing $\overline{x}_2(y_1) = \frac{y_1-c}{2}$, we have

$$\begin{aligned}
U_2(y_1) =& \int_0^{\overline{x}_2(y_1)} \left[v_2(x_2) + (\overline{x}_2(y_1) - x_2)^2\right]dx_2 + \int_{\overline{x}_2(y_1)}^{y_1} v_2(x_2)dx_2 + \overline{x}_2(y_1)^2(1-y_1)\\
=& \int_0^{y_1} v_2(x_2)dx_2 + \int_0^{\overline{x}_2(y_1)} (\overline{x}_2(y_1) - x_2)^2 dx_2 + \overline{x}_2(y_1)^2(1-y_1)\\
=& \frac{1}{2}y_1^2 - \frac{1}{6}y_1^3 + (1-y_1)\overline{x}_2(y_1)^2 + \frac{1}{3}\overline{x}_2(y_1)^3,\\
\Psi_2(y_1) =& \int_0^{y_1} v_2(x_2)dx_2 + \int_0^{\overline{x}_2(y_1)} [v_2(y_1-c-x_2) - c(y_1-c-2x_2) - v_2(x_2)]\,dx_2\\
&+ [v_2(1-c) - c(1-c)](1-y_1)\\
=& \int_0^{y_1} v_2(x_2)dx_2 + \int_0^{\overline{x}_2(y_1)} \frac{1}{2}(2-c-y_1)(y_1-c-2x_2)\,dx_2 + \frac{1}{2}(1-c)^2(1-y_1)\\
=& \int_0^{y_1} v_2(x_2)dx_2 + \frac{1}{2}(2-c-y_1)\overline{x}_2(y_1)^2 + \frac{1}{2}(1-c)^2(1-y_1)\\
=& \frac{1}{2}y_1^2 - \frac{1}{6}y_1^3 + (1-y_1)\overline{x}_2(y_1)^2 + \overline{x}_2(y_1)^3 + \frac{1}{2}(1-c)^2(1-y_1), \text{ and}\\
\Pi_2(y_1) =& \Psi_2(y_1) - U_2(y_1) = \frac{2}{3}\overline{x}_2(y_1)^3 + \frac{1}{2}(1-c)^2(1-y_1).
\end{aligned}$$

(c) When $y_1 \in [1, 2-c]$, noticing $\overline{x}_2(y_1) = \frac{y_1 - c}{2} \leq 1 - c$ and $v_2(x_2) = \frac{1}{2}$ for $x_2 \geq 1$, we have

$$\begin{aligned}
U_2(y_1) &= \int_{y_1-1}^{y_1} v_2(x_2)dx_2 + \int_{y_1-1}^{\overline{x}_2(y_1)} (\overline{x}_2(y_1) - x_2)^2 dx_2 \\
&= \int_{y_1-1}^{1} (x_2 - \frac{1}{2}x_2^2)dx_2 + \int_{1}^{y_1} \frac{1}{2}dx_2 + \int_{y_1-1}^{\overline{x}_2(y_1)} (\overline{x}_2(y_1) - x_2)^2 dx_2 \\
&= \left(\frac{1}{2}x_2^2 - \frac{1}{6}x_2^3\right)\Big|_{y_1-1}^{1} + \frac{1}{2}(y_1 - 1) + \frac{1}{3}\left[\overline{x}_2(y_1) - (y_1 - 1)\right]^3 \\
&= \frac{1}{3} + \frac{1}{2}(y_1 - 1) - \frac{1}{2}(y_1 - 1)^2 + \frac{1}{6}(y_1 - 1)^3 + \frac{1}{24}(2 - c - y_1)^3, \\
\Psi_2(y_1) &= \int_{y_1-1}^{y_1} v_2(x_2)dx_2 + \int_{y_1-1}^{\overline{x}_2(y_1)} \left[v_2(y_1 - c - x_2) - c(y_1 - c - 2x_2) - v_2(x_2)\right] dx_2 \\
&= \int_{y_1-1}^{y_1} v_2(x_2)dx_2 + \int_{y_1-1}^{\overline{x}_2(y_1)} \frac{1}{2}(2 - c - y_1)\left(y_1 - c - 2x_2\right) dx_2 \\
&= \int_{y_1-1}^{y_1} v_2(x_2)dx_2 + \frac{1}{2}(2 - c - y_1)\left[\overline{x}_2(y_1) - (y_1 - 1)\right]^2 \\
&= \frac{1}{3} + \frac{1}{2}(y_1 - 1) - \frac{1}{2}(y_1 - 1)^2 + \frac{1}{6}(y_1 - 1)^3 + \frac{1}{8}(2 - c - y_1)^3, \text{ and} \\
\Pi_2(y_1) &= \Psi_2(y_1) - U_2(y_1) = \frac{1}{12}(2 - c - y_1)^3.
\end{aligned}$$

$\blacksquare$