Statement of Research Interests

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My current research interests are mostly twofold and concern the areas of meromorphic / rational approximation and random polynomials/random matrix theory. The next section offers a general description of the questions I am interested in and the subsequence ones provide a more detailed description of particular works.

Summary

The asymptotic theory of meromorphic and rational approximation to functions holomorphic at infinity is primarily concerned with establishing the types of convergence, describing the domains where this convergence takes place, and identifying its exact rates. As the first question is classical, it is the latter two topics that my work is mostly focused on with the greater emphasis on the exact rates. Three groups of approximants are introduced: meromorphic (AAK-type) approximants, L^2-best rational approximants, and rational interpolants with free poles. Despite the groups being distinctively different, they share one common feature: much of the information on their asymptotic behavior is encoded in the weighted non-Hermitian orthogonality relations satisfied by the polynomials vanishing at the poles of the approximants with the weight of orthogonality coming from the approximated function. The main goal of the study is extracting the generic asymptotic behavior of the zeros of these polynomials from the orthogonality relations and tracking down those zeros that do not conform to the general pattern (wandering poles of the approximants).

Besides pure mathematical interest, this work is motivated by the practical question of solving the inverse EEG(ElectroEncephaloGraphy) problem. The inverse problem arising in EEG consists in estimating neural current sources located within the brain from the outside measurements. Roughly speaking, in the innermost layer (the brain), there may be singularities due to the presence of current sources. These singularities are to be localized from the available data on the outer boundary (the scalp). When the domain is a spherical ball, the above issue is equivalent to a sequence of 2D inverse problems, each of which consists of recovering the singularities of some algebraic function in a disk from the knowledge of the function on the circumference. The latter problem falls into the category of "crack detection" problems and is asymptotically solved by approximating the given algebraic function by meromorphic ones while observing the dynamics of their poles.

I am also interested in describing the eigenvalue statistics for ensembles of random normal matrices, or equivalently, statistics of the charged particles in the plane that interact in the following way. Imagine a collection of N particles of unit charge confined to the plane and interacting so that the contribution to the potential energy of the system by a pair of particles is proportional to the logarithm of the distance between them. These particles are placed in the presence of an external field which is repulsive at infinity. The entire system is in contact with a heat reservoir so that the temperature remains constant. The energy of the total system (electrostatic + heat bath) is constant, but the energy of the electrostatic system is not, and the likelihood of the system being in a state with energy E is proportional to $e^{-\beta E}$ where, β is a dimensionless quantity representing the inverse temperature. The external field is assumed to be zero on some compact set, harmonic off this set, and creating repulsion at infinity as if a particle of charge $s > N$ was placed there. Amazingly (from an analytical point of view), such densities also arises in number theory while investigating the volume of polynomials of bounded Mahler measure. Here we look at polynomials, either with complex or only real coefficients (even though it is an innocuously looking distinction, the theories are strikingly different) of degree N whose Mahler measure is bounded by a certain fixed constant. In the classical case the Mahler measure is nothing but the absolute value of the product of those zeros of the polynomial that lie outside of the unit disk times the leading coefficient the polynomial. The coefficients of such polynomials form a certain volume in $(N+1)$-dimensional space (either complex or real). We select a polynomial at random from this volume with respect to the uniform measure and want to know the expected behavior of the zeros of such a polynomial. As it turns out, mathematically, the questions about the behavior of the zeros of these polynomials are no different from the questions about the behavior of the interacting particles / eigenvalues of normal matrices in logarithmic external fields described above.

Asymptotics of Padé approximants for functions with branch points

Even though "most" of the numbers are irrational or even transcendental, the numbers that we, and definitely our computer systems, can deal with are rational. By now it is well understood that to get a "good" (the error of approximation is small relatively the size of the denominator of the approximant) approximation of an irrational number by a rational number, one needs to develop an irrational number into an infinite continued fraction and then take a finite section of this fraction. An analogous procedure for power series leads to rational interpolants that are called Padé approximants. Such approximants can be characterized in the following way: given a convergent power series, say at infinity,

$$f(z) = \sum_{i=1}^{\infty} f_i z^{-i},$$

the n-th classical diagonal Padé approximant to the function $f(z)$ is a rational function $[n/n]_f(z) = P_n(z)/Q_n(z)$ such that $\deg(P_n), \deg(Q_n) \leqslant n$ and the difference $Q_n(z)f(z) - P_n(z)$ vanishes at infinity with order at least $n+1$. If $\deg(Q_n)$ is exactly n, then the difference $f(z) - [n/n]_f(z)$ vanishes at infinity with order at least $2n+1$. Such a closeness between $f(z)$ and $[n/n]_f(z)$ suggests that it should hold in a large domain. Convergence theory of Padé approximants consists in identifying these domains and proving the convergence of the approximants to the approximated function.

It is well known that the least approximable irrational numbers are algebraic. Hence, an important and interesting question in Padé approximation is the convergence of approximants to algebraic functions $f(z)$ (these are functions that can be holomorphically extended from infinity along any path in the complex plane not passing through a finite number of points and at those points the continuations are not allowed to have essential singularities). Padé approximants as rational functions are single-valued (they assign to each point in the plane a unique value). Algebraic functions on the other hand are multi-valued, since distinct paths from infinity to a given point can lead to distinct continuations (continuation depends on how the path winds around the singularities of the function). Thus, Padé approximants cannot converge to algebraic $f(z)$ everywhere outside of its singularities. However, if the singularities of $f(z)$ are connected by a compact set in such a way that no path from infinity can wind around them to produce distinct continuations, $f(z)$ is reduced to a single-valued branch in the complement of this compact set. It was Herbert Stahl who, given an algebraic function $f(z)$, identified the unique compact set, the contour of minimal capacity, outside of which the approximants do converge to $f(z)$.

In Stahl's work the allowed set of singularities was in fact infinite (but polar) and the convergence was shown in "weak" (n-th root) sense. In [1] the convergence is proven in the "strong" sense, but it is assumed that the set of singularities is finite and a further assumption on the symmetric contours is placed (the assumptions hold "generically", that is, for most of the contours). In [3] this generic assumption on the contours was removed at the expense of restricting the type of branching at singularities (only quadratic branching was allowed). My graduate student A. Barhoumi works on removing the "genericness" assumption in [1].

Classical Padé approximants can be replaced by the multipoint Padé approximants, which are rational functions $P_n(z)/Q_n(z)$, $\deg(P_n), \deg(Q_n) \leqslant n$, that interpolate $f(z)$ at system of $2n+1$ pre-assigned possibly coincidental points (for classical Padé approximants all the interpolation points lie at infinity). In [4], the questions of existence and geometry of convergence domains were considered as well as of the convergence in "strong" sense.

One advantage of multipoint Padé approximants is that they can interpolate several distinct functions simultaneously. When one interpolates at m distinct points m distinct functions (one point for each function), it was shown by Buslaev that the plane is partitioned into m domains (one domain for each interpolation point and the corresponding function is analytic in this domain) such that the multipoint Padé approximants simultaneously converge to the approximated functions in these domains. Convergence was shown to hold in "weak" sense (this is a generalization of Stahl's work). In [2], convergence in the "strong" sense was shown to hold when $m = 2$ for a certain class of approximated functions.

References

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Strong asymptotics of Hermite-Padé approximants

Hermite-Padé approximants are vector generalizations of Padé approximants, and were introduced by Hermite as a tool in proving that the number e is transcendental. Up to this day, number theorists are using them to prove transcendence of certain values of the zeta function. Exactly as in the case of Padé approximants, they relate to orthogonal, in this case multiple orthogonal, polynomials. The set up of [2] is as follows. Let $[a_1, b_1]$ and $[a_2, b_2]$ be two disjoint intervals on the real line (in fact, [2] deals with an arbitrary number of such intervals). Fix two integers n_1 and n_2 and consider the polynomial $Q_{n_1,n_2}(x) = x^{n_1+n_2} + \dots$ such that

$$0 = \int_{a_i}^{b_i} Q_{n_1,n_2}(x) x^k f_i(x) dx, \quad \text{for any} \quad k \in \{1, \dots, n_i - 1\},$$

where $f_i(x)$ is a certain function on the interval $[a_i, b_i]$. In [2], the asymptotic behavior of $Q_{n_1,n_2}(x)$ is investigated when $n_1, n_2 \to \infty$ and $f_1(x), f_2(x)$ are Fisher-Hartwig perturbations of functions holomorphic around $[a_1, b_1], [a_2, b_2]$, respectively. A Riemann-Hilbert characterization of multiple orthogonal polynomials and the steepest descent analysis of the corresponding Riemann-Hilbert problem are used. The main technical difficulty of the above method lies in constructing so-called local parametrices. In [2], these are built out of matrices solving the Riemann-Hilbert problem characterizing solutions of the Painlevé XXXIV equation. As part of the proof, the asymptotic behavior of these matrices with respect to a certain parameter is analyzed.

In [1], an analogous question was studied under the assumption that $[a_1, b_1] = [-1, a]$ and $[a_2, b_2] = [-a, 1]$, $a \in (-1, 1)$. That is, the intervals are no longer disjoint but overlapping. "The geometry" of this problem is significantly more complicated. In [1], the main objects describing the asymptotics of the polynomials $Q_{n_1,n_2}(x)$ came from a certain easily constructed Riemann surface of genus zero. Construction of the Riemann surface for the set up in [1] was much more complicated and the surface ended up having genus either 1 or 2 depending on the value of a. On the technical side, the steepest descent analysis of the corresponding Riemann-Hilbert problem again used in [1], was more involved, in particular, one needed to construct not only local, but also global "lenses".

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Expected number of real zeros of certain random polynomials

Let $p(x)$ be a polynomials of degree n with real coefficients, that is,

$$p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0, \quad a_i \in (-\infty, \infty).$$

Equivalently, we can write $p(x) = a_n \prod_{i=1}^{n}(x - x_i)$ where the zeros x_i are either real or come in conjugate symmetric pairs. Different sets of coefficients $\{a_n, \ldots, a_0\}$ correspond to different sets of zeros $\{z_1, \ldots, z_n\}$. In particular, the number of real zeros changes as well. Assume now that the coefficients $\{a_n, \ldots, a_0\}$ are real independent identically distributed random variables. One might ask then what is the expected number of real zeros of such a polynomial? When the law for the coefficients is Gaussian, Kac has shown that the expected number is $\frac{2}{\pi}\log(n+1)$ plus some bounded correction.

Monomials z^k can be interpreted in many ways. In particular, they are orthogonal to each other with respect to the arclength measure on the unit circle. What if we then rewrite a random polynomial in a different basis, that is,

$$p_n(x) = a_n\phi_n(x) + a_{n-1}\phi_{n-1}(x) + \cdots + a_0\phi_0(x),$$

where polynomials $\phi_k(x)$ have degree k and real coefficients, and are orthogonal on the unit circle with respect to some measure, say μ. The following question was investigated in [2]. Which conditions placed on the measure μ guarantee that the expected number of real zeros is still of order $\frac{2}{\pi}\log(n+1)$ when the coefficients $\{a_n, \ldots, a_0\}$ are chosen according to the Gaussian law? An answer was provided in terms of the speed of decay of the recurrence coefficients corresponding to μ.

For Kac polynomials not only the leading order $\frac{2}{\pi}\log(n+1)$ for the expected number of real zeros is known, but also a the existence of a full asymptotic expansion $\frac{2}{\pi}\log(n +$

$1) + \sum_{i=1}^{\infty} A_i(n+1)^{-i}$. In [1], which is a joint work with my graduate student, such an expansion is obtained when the underlying measure μ is absolutely continuous with respect to the arclength measure and the Radon-Nikodym derivative extends to a holomorphic non-vanishing function in some neighborhood of the unit circle.

References

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Root statistics of random polynomials with bounded Mahler measure

Once again, write a polynomial of degree n with real coefficients as $p(x) = a_n \prod_{i=1}^{n}(x - x_i)$. The λ-homogeneous Mahler measure of $p(x)$ is given by

$$M(p) = |a_n|^{\lambda} \prod_{i=1}^{n} \max\{1, |x_i|\}.$$

The degree n Mahler unit star body is defined by

$$B_{n,\lambda} = \{p(x) : \deg(p) = n \text{ and } M_{\lambda}(p) \leqslant 1\}.$$

Identifying a polynomial $p(x)$ with the vector of its coefficients, $B_{n,\lambda}$ can be identified with a certain (finite) volume in $n+1$-dimensional (real) space. There are long-standing open questions in number theory concerning zeros of polynomials in $B_{n,\lambda}$. What is investigated in [1] is the behavior of not particular but rather "average" polynomials in $B_{n,\lambda}$.

More precisely, choosing a vector at random from the volume corresponding to $B_{n,\lambda}$ (with respect to uniform density) produces a random polynomial and the zeros of this polynomial form a random configuration of n points in the complex plane with some points belonging to the real line and the rest coming in pairs symmetric with respect to this axis. Given a set on the real line and/or in the upper half-plane, we would like to count how many of the points (zeros) belong to each of these sets. These numbers are again random variables. It was shown by Sinclair that expected values of these random variables can be computed using Pfaffians of a certain 2×2 matrix kernel $\mathbf{K}_{n,\lambda}$. In [1], which is a joint work with Sinclair, we analyze the asymptotic behavior of $\mathbf{K}_{n,\lambda}$ as n becomes large for all possible choices of sets on the real line and the upper half plane.

A polynomial $p(x) = a_n \prod_{i=1}^{n}(x - x_i)$ is called reciprocal if $1/x_j$ is its zero whenever x_j is its zero. It is worth noting that one of the long standing questions in number theory alluded to in the first paragraph is resolved for non-reciprocal polynomials. In [2], questions raised in [1] were studied for reciprocal polynomials.

References

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