Pricing a Collateralized Debt Obligation

- A Collateralized Debt Obligation (CDO) is a structured financial transaction; it is one kind of Asset-Backed Security (ABS).

- A Manager designs a portfolio of debt obligations, such as corporate bonds (in a CBO) or commercial loans (in a CLO).

- The Manager recruits a number of investors who buy rights to parts of the portfolio.

- Each investor buys the right to receive certain cash flows derived from the portfolio, divided into *tranches*.

- The *senior tranche* has first call on the cash flows (interest and principal), up to a set percentage.

- The *junior tranche* has next call, again up to a set percentage.

- Remaining cash flows are passed through to the *equity tranche*.

- Simplified example: suppose that the portfolio consists of two corporate bonds, B_1 and B_2, and neither pays interest.

- The bonds are priced at b_1 and b_2 at $t = 0$.

- Each bond returns 1 at $t = T$ if its issuer is not in default, and 0 if the issuer has defaulted.

- The matrix $\mathbf{D}$ is:

In default:	Neither	Issuer 1	Issuer 2	Both
Cash	e^{rT}	e^{rT}	e^{rT}	e^{rT}
B_1	1	0	1	0
B_2	1	1	0	0

- $\mathbf{D}$ is 3×4, so $N = 3 < n = 4$, and the market is not complete.

- Now

$$\mathbf{S}_0 = \begin{pmatrix} 1 \\ b_1 \\ b_2 \end{pmatrix}.$$

- If $0 < b_i < e^{-rT}, i = 1, 2$, then $\mathbf{S}_0 = \mathbf{D}\boldsymbol{\psi}$ where

$$\boldsymbol{\psi} = e^{-rT} \begin{pmatrix} (1 - p_1)(1 - p_2) \\ p_1(1 - p_2) \\ (1 - p_1)p_2 \\ p_1 p_2 \end{pmatrix}$$

and $p_i = 1 - e^{rT} b_i, i = 1, 2$.

- Clearly $\boldsymbol{\psi}$ is a state price vector, and the corresponding risk-neutral measure $\mathbb{Q}$ implies that the probabilities of default are p_1 and p_2.

- It also implies independence of the events of default.

- But because the market is not complete, other state price vectors and other risk-neutral measures exist.

- Suppose that the CDO has just a senior tranche S and the equity tranche E, and each receives 50% of the cash flows (which are only the return of principal at $t = T$).

- With these added to the market, $\mathbf{D}$ becomes

In default:	Neither	Issuer 1	Issuer 2	Both
Cash	e^{rT}	e^{rT}	e^{rT}	e^{rT}
B_1	1	0	1	0
B_2	1	1	0	0
S	1	1	1	0
E	1	0	0	0

- Actually, one of S and E is redundant, because $S + E = B_1 + B_2$. For convenience, we drop E.

- If the senior tranche has price s at $t = 0$, we find

$$\boldsymbol{\psi} = \mathbf{D}^{-1}\mathbf{S}_0 = \begin{pmatrix} b_1 + b_2 - s \\ s - b_1 \\ s - b_2 \\ e^{-rT} - s \end{pmatrix}.$$

- For the market to be arbitrage-free, we must have

$$\max(b_1, b_2) < s < \min\Big(e^{-rT}, b_1 + b_2\Big).$$

- Note that seniority means that S costs more than either bond.

- If $\mathbb{Q}_s[\cdot]$ is the corresponding risk-neutral measure, we find that the probability of default of issuer 1 is

$$e^{rT}\left(s - b_1 + e^{-rT} - s\right) = 1 - e^{rT} b_1 = p_1,$$

as before, and similarly p_2 for issuer 2.

- But the probability that they both default is $1 - se^{rT}$, and this equals $p_1 p_2$ only when

$$s = b_1 + b_2 - e^{rT} b_1 b_2.$$

- Typically, $s < b_1 + b_2 - e^{rT} b_1 b_2$, which means that the probability of both issuers defaulting is higher than it would be under independence.

- In this case, there is *positive dependence* between the events of default.

- Dependence is often modeled using a *Gaussian copula*.

- Suppose that default is associated with random variables Z_1 and Z_2, normally distributed with mean 0 and variance 1, and correlation ρ.

- Issuer 1 defaults if and only if $Z_1 < \Phi^{-1}(p_1)$, and similarly issuer 2.

- The probability that *both* default is a function of ρ.

- If $\rho = 0$, events of default are independent.

- The value of ρ that corresponds to the risk-neutral probability of both issuers defaulting is the *implied correlation*.