Risk Management for Private Equity Funds

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Abstract

Although risk management has been a well-ploughed field in financial modeling for over three decades, the understanding how to correctly quantify and manage the risks of investing in private equity remains limited and continues to considerably lag that of other traditional asset classes. The objective of this paper is to fill this gap by developing the first comprehensive risk management framework for private equity fund investments. The framework captures the three main sources of risks that private equity investors face when investing in the asset class: *market risk*, *liquidity risk*, and *cash flow risk*. Underlying the framework is a stochastic model for the value and cash flow dynamics of private equity funds, which allows to derive three dynamic risk measures for private equity fund investments: Value-at-Risk (*VaR*), Liquidity-Adjusted Value-at-Risk (*LVaR*), and Cash-Flow-at-Risk (*CFaR*). The model is calibrated to historical data of buyout funds and the dynamics of the developed risk measures are illustrated using Monte-Carlo simulations. A sensitivity analysis shows the impact of changes in the main model parameters on risk measures.

Keywords: Private equity, risk management, Value-at-Risk, Cash-Flow-at-Risk, Liquidity-Adjusted Value-at-Risk.

JEL Classification: G17, G23, G24

Private equity continues to grow in importance as an asset class as investors seek diversification benefits relative to traditional stock and bond holdings. Especially large institutional investors like insurance companies, endowment and pension funds allocate increasing portions of their overall investment portfolios to private equity. The vast majority of these investments is intermediated through funds, because entering, managing, and exiting direct private equity investments requires high levels of expertise and experience. Despite the increasing importance of private equity as an asset class, our understanding how investors can adequately measure and manage the risks of these investments remains limited and continues to considerably lag that of other traditional asset classes. The aim of this paper is to fill this gap by developing the first comprehensive risk management framework for private equity fund investments.

Private equity funds have at least two key institutional features that differentiate them from traditional investments into traded stocks or bonds and make risk management a challenging task. First, private equity fund investments are illiquid and long-term. Private equity funds typically have maturities of ten to 14 years and secondary markets for private equity fund positions are still highly inefficient, making it costly for investors to sell their positions. Second, private equity fund investments involve specific dynamics of capital drawdowns and distributions. The private equity fund investor first makes an initial capital commitment and, at a later time, transmits specific amounts of capital to the general partner in response to capital calls (or capital drawdowns). The timing and size of capital calls are not known until they are announced, and usually there is a substantial lag between the time at which capital is committed to a fund and the time at which that capital is actually drawn for investment. In addition, cash payouts (or capital distributions) of a private equity fund are also uncertain, while these payouts are significant, because of the bounded lifecycle of the funds. Thus, the invested capital changes dynamically over the lifetime of a fund and private equity fund investments require active cash flow management of capital calls and distributions.

Taking into account these special features, a risk management framework for private equity fund investments has to capture three main sources of risk: (i) *Market Risk*: The risk of losses in the market prices of the portfolio companies held by a fund exposes investors to market risk; (ii) *Liquidity Risk*: The illiquidity of private equity partnership interests exposes investors to asset liquidity risk associated with selling positions in the secondary markets at potentially large and ex-ante uncertain discounts on a fund's net-asset-value; (iii) *Funding (or Cash Flow) Risk*: The unpredictable timing and magnitude of fund cash flows poses funding and cash flow risks to investors. In particular, capital commitments are contractually binding and defaulting on these payments can result in the loss of the entire private equity partnership interest.

The framework developed in this paper allows addressing these three main sources of risk using distinct risk measures. Underlying the risk management framework is a model of the dynamics of private equity funds that consists of three main components, which correspond to the essential phases of the private equity fund lifecycle:[1] the drawdowns from the committed capital paid into the fund; the performance of the investments effected by the fund; and the distributions of dividends and proceeds taken out of the fund. Effectively, a standard lognormal process that is correlated with aggregate stock market returns is used for the dynamics of the investment value. Capital drawdowns and distributions are modeled by two stochastic processes for the dynamics of the rates of drawdowns and repayments. Taking into account that the speed of capital drawdowns and distributions may be affected by the overall state of the economy (or stock market), the model also incorporates the important possibility that the drawdown and repayment rates are correlated with aggregate stock market returns. Given this stochastic model, the paper proposes three risk measures for private equity fund investments.

Market risk is captured by using a Value-at-Risk (*VaR*) approach, which has become the standard measure that financial analysts use to quantify this risk (see Jorion (2001)).[2]

[1]The model used is a continuous-time stochastic version of the deterministic model developed by Takahashi and Alexander (2002).

[2]Besides its risk management applications, *VaR* measures are also important for regulatory capital requirements. In particular, the Basel Committee on Banking Supervision at the Bank for International Settlements imposes to financial institutions such as banks and investment firms to meet capital

Generally, *VaR* is defined as the maximum potential loss in value of a portfolio of financial instruments with a given probability over a certain horizon. In simpler words, it is a number that indicates how much a financial institution can lose with a given probability over a given time horizon. Standard *VaR* implementations assume a static setting in which the mix of a position is unchanged over time, i.e., no cash inflows or outflows occur over the *VaR* horizon. In contrast, the *VaR* approach presented here is dynamic in the sense that it also takes into account that the investor's risk exposure varies over the lifecycle of the fund with stepwise capital drawdowns and distributions.

The *VaR* measure developed is based on the implicit assumption that an investor can sell his position in the fund at any time at the fund's current net-asset-value. In reality, stakes in private equity funds are highly illiquid and can typically only be sold at some discount on the secondary private equity markets. In order to account for this form of *liquidity risk*, the *VaR* calculation is also extended to a Liquidity-Adjusted Value-at-Risk (*LVaR*). The main idea of this *LVaR* is to include secondary markets discounts as an exogenous liquidity cost in the *VaR* calculation. Capturing the unpredictable nature of secondary market discount dynamics, discounts are thereby modeled using a mean-reverting Ornstein-Uhlenbeck process, which is also assumed to be correlated with aggregate stock market returns.

Finally, to capture *funding (or cash flow) risk*, the paper introduces a Cash-Flow-at-Risk (*CFaR*) measure. This measure is defined as the change (loss) in the investor's cash position, which is only exceeded with some given probability, over a given time horizon. This measure is also dynamic and its interpretation changes with the stage in the lifecycle of a fund, as discussed in more detail below.

After calibrating the model such that parameters correspond to an investment in a typical buyout fund, the developed risk measures are illustrated by using Monte-Carlo simulations. The results underline that there exists private equity specific patterns in *VaR* dynamics that a sensible risk management framework for fund investors must take

requirements based on *VaR* estimates.

into account. In particular, the *VaR* for some fixed time horizon initially rises sharply but then peaks and decreases to zero over the fund lifetime. This behavior is consistent with the typical lifecycle of private equity funds. As the fund gradually draws down capital and builds up the investment portfolio, the investor's risk exposure and consequently also the maximum possible loss that can occur with some given probability both increase markedly. After the maximum level has been reached, the fixed horizon *VaR* decays rapidly towards zero as capital distributions of the fund stepwise decrease the investor's risk exposure. The simulation results also show that the maximum value of the fixed horizon *VaR* is large but well below the investor's initial capital commitment for all confidence levels considered. A main reason for this is that stepwise capital drawdowns and distributions of private equity funds lead to a situation where the investor's committed capital is typically never fully invested in a fund.

The comparison between the simulated *VaR* and *LVaR* measures highlights that the effects of illiquidity on the investor's risk exposure can be substantial over the life of a fund. However, the numerical results also add that illiquidity only increases the investor's risk exposure if the time horizon under consideration is smaller than the fund's total remaining lifetime, as liquidity risk is fully resolved with the final liquidation of a fund. The effects of liquidity risk are thus investor specific. For investors that face the risk of being forced to sell their private equity positions on the secondary markets due to a surprise liquidity shock during the fund lifetime, ignoring liquidity risk can lead to substantially underestimated risk figures. In contrast, for 'deep-pocket' investors with sufficient cash reserves and a low probability of facing surprise liquidity shocks, liquidity risk is only of minor importance.

The dynamics of the simulated *CFaR* measure further illustrate how cash flow risk changes with the development stage of the fund. In the early stages of a fund's life, when the fund starts to call capital from the investors, *CFaR* is large and gives a measure of funding risk, i.e., the maximum amount of cash that an investor must hold in reserve in order to being able to meet capital calls with a given probability. In the later times of a fund's life, when the funds starts to gradually exit its portfolio companies and returns the proceeds to the investors, *CFaR* decreases and gives a measure of distribution risk, i.e., the negative value of *CFaR* gives an indication of the minimum amount of cash distributed by the private equity fund over some given time horizon with some given probability.

This paper is related to studies that evaluate the risk and return characteristics or cash flow dynamics of private equity investments. Important research in this area includes, among others, Cochrane (2005), Kaplan and Schoar (2005), Ljungqvist et al. (2008), Phalippou and Gottschalg (2009), Korteweg and Sorensen (2010), Robinson and Sensoy (2011), Driessen et al. (2012), Ang et al. (2013), Harris et al. (2013), Higson and Stucke (2013), Ewens et al. (2013), Buchner and Stucke (2014), and Hochberg et al. (2014). None of these studies, however, develops an integrated risk model for private equity investments. The only scientific paper in the area of risk management for private equity investments is a study by Bongaerts and Charlier (2009), who apply existing credit risk models to individual private equity investments.[3] However, their model is difficult to calibrate and lacks to address liquidity and cash flow risks of private equity fund investments. I am unaware of any other existing study that develops a risk model for private equity investments that captures market, liquidity, and cash flow risk using standard *VaR* measures. Developing a model that is sufficiently tractable to accurately quantify these risks is the main contribution of this paper.

The remainder of this paper is organized as follows. Section 1 presents the model of the cash flow and value dynamics of private equity funds. Section 2 develops the three risk measures. Section 3 shows how these risk measures can be estimated using Monte-Carlo simulations. Section 4 presents the results of the model calibration and of the numerical model analysis. Section 5 concludes.

[3]In addition, there are several papers by industry analysts on the issue of risk management, like Weidig (2002) or Diller and Herger (2008). Most of these papers simply perform ad-hoc bootstrap simulations of fund IRRs or multiples using historical cash flow data.

1 The Model

This section develops the model used to derive the risk measures of private equity funds. The model is formulated in a continuous-time framework. It is assumed that all random variables introduced in the following are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and that all random variables indexed by t are measurable with respect to the filtration $\{\mathcal{F}_t\}$, representing the information commonly available to all investors.

1.1 Private Equity Fund Dynamics

Following the typical construction of private equity funds, modeling the dynamics of private equity funds requires three main components: the modeling of the investment value, of the capital drawdowns, and of the return repayments.

A. Investment Value

The first step in modeling private equity funds is characterizing value dynamics. Consider a fund with a total legal lifetime given by T_l. Let V_t denote the value of the fund's investment portfolio at time t, cumulated capital drawdowns from the LPs up to time t are represented by D_t, and cumulated capital distributions to the LPs up to time t are represented by R_t. To keep things simple, it is assumed that the return on any cash flow invested in the fund can be described by a normal distribution with constant mean, constant volatility, and a constant correlation with aggregate stock market returns.

Assumption 1.1 *The dynamics of the fund value, V_t, under the objective probability measure $\mathbb{P}$ can be described by the stochastic process $\{V_t, 0 \leq t \leq T_l\}$:*

$$dV_t = V_t(\mu_V dt + \beta_V \sigma_M dB_{M,t} + \sigma_\epsilon dB_{\epsilon,t}) + dD_t - dR_t, \tag{1.1}$$

where $\mu_V > 0$ is the mean rate of return, $\sigma_M > 0$ is the return volatility of the aggregate stock market, β_V is the market beta of the fund, and $\sigma_\epsilon > 0$ is idiosyncratic volatility;

$B_{M,t}$ is a standard Brownian motion driving aggregate stock market returns[4] *and $B_{\epsilon,t}$ is a second Brownian motion, representing idiosyncratic shocks, i.e., $dB_{M,t}dB_{\epsilon,t} = 0$.*

This specification of the value dynamics is straightforward. The first term on the right hand side of the equation, $V_t(\mu_V dt + \beta_V \sigma_M dB_{M,t} + \sigma_\epsilon dB_{\epsilon,t})$, states that the instantaneous change in value of a private equity fund is made of the performance of the investment already in place at time t. Investments in place are assumed to have normally distributed returns with constant mean (after management fees and carry payments) given by μ_V and a total variance given by $\sigma_V^2 = \beta_V^2 \sigma_M^2 + \sigma_\epsilon^2$, where $\beta_V^2 \sigma_M^2$ represents systematic variance and σ_ϵ^2 is idiosyncratic variance. This specification allows for the important possibility that private equity fund investment returns and aggregate stock market returns are (potentially highly) correlated with a coefficient of correlation given by $\rho_V = \beta_V \sigma_M / \sigma_V$.

The second and third term on the right hand side of (1.1) show that fund values are augmented by instantaneous capital drawdowns, dD_t, and decreased by instantaneous capital distributions, dR_t. Including dD_t and dR_t into equation (1.1) takes into account that private equity fund investments typically involve stepwise capital drawdowns and generate substantial intermediate capital distributions during the bounded fund lifecycle. Note that capital distribution (whether in form of cash or marketable securities) are directly distributed to the investors as a fund gradually exits its investments. Therefore, capital distributions simply decrease the fund value V_t, and there is no need to impose any assumption on the reinvestment of intermediate cash flows.

An important additional feature of the model is that it allows a fund to earn a risk-adjusted excess return, called alpha. Formally, this excess return is defined as:

$$\alpha_V = \mu_V - r_f - \beta_V(\mu_M - r_f), \tag{1.2}$$

where $\mu_M > 0$ is the mean rate of return of the aggregate stock market and r_f is the

[4] It is implicitly assumed here that aggregate stock market dynamics, M_t, can be described by a standard geometric Brownian motion given by: $dM_t = \mu_M M_t dt + \sigma_M M_t dB_{M,t}$, with $\mu_M > 0$ being the mean rate of return, and $\sigma_M > 0$ being the return volatility.

constant riskless rate. Intuitively, this alpha is the risk-adjusted (net) excess return of the portfolio companies owned and managed by the private equity fund. This excess return may arise from improved governance or from the GP being a skillful manager (see, e.g. Sorensen et al. (2013)).

B. Capital Drawdowns

To model capital drawdowns, it is assumed that the fund has an initial committed capital given by C_0 and a total commitment period (i.e., the period over which capital drawdowns can occur) given by T_c (with $T_c \leq T_l$). I assume that capital is drawn at time t at some stochastic rate δ_t from the remaining undrawn committed capital, i.e., $C_0 - D_t$.

Assumption 1.2 *The dynamics of the cumulated capital drawdowns, D_t, can be described by the ordinary differential equation*

$$dD_t = \delta_t(C_0 - D_t)1_{\{0 \leq t \leq T_c\}}dt, \tag{1.3}$$

where $1_{\{\cdot\}}$ is an indicator function. The fund's drawdown rate δ_t is assumed to follow a stochastic process $\{\delta_t, 0 \leq t \leq T_c\}$ with specification given by

$$\delta_t = \delta + \sigma_\delta B_{\delta,t}, \tag{1.4}$$

where $\delta > 0$ is the mean of the drawdown rate and $\sigma_\delta > 0$ reflects the volatility of the drawdown rate; $B_{\delta,t}$ is a third standard Brownian motion for which it is assumed that $dB_{\delta,t}dB_{M,t} = \rho_\delta dt$ and $dB_{\delta,t}dB_{\epsilon,t} = 0$.

Equation (1.3) is essentially a continuous-time version of the deterministic discrete-time dynamics introduced by Takahashi and Alexander (2002) for the capital drawdowns of private equity funds. Malherbe (2004) uses a similar continuous-time specification, but employs a different stochastic process for the drawdown rate dynamics.

In most cases, capital drawdowns of private equity funds are concentrated in the first few years (or even quarters) of a fund's life. After high initial investment activity, drawdowns of private equity funds are carried out at a declining rate, as fewer new investments are made, and follow-on investments are spread out over a number of years. This typical time-pattern is well reflected in the structure of equation (1.3), which implies that initially high capital drawdowns at the start of a fund decrease exponentially over the commitment period T_c. In addition, capital drawdowns are ... Under this modeling framework, cumulated drawdowns D_t can also never exceed the total committed capital C. At the same time the model allows for a certain fraction of the committed capital C not to be drawn, as the commitment period T_c acts as a cut-off point for capital drawdowns. Finally, the model also incorporates the important possibility that the drawdown speed has a non-zero correlation ρ_δ with aggregate stock market returns.

C. Capital Distributions

The final step in modeling private equity funds is characterizing the dynamics of the capital distributions. Recognizing that the size and timing of repayments depend on the performance of the fund's investment portfolio, it is assumed that capital distributions at time t occur at a stochastic rate ν_t from the total investment portfolio value V_t of the fund.

Assumption 1.3 *The dynamics of the cumulated capital distributions, R_t, can be described by*

$$dR_t = \nu_t V_t dt, \quad \textit{for} \quad t < T_l, \quad \textit{and} \quad R_t = \int_0^t \nu_u V_u du + V_t 1_{\{t=T_l\}}, \quad \textit{for} \quad t \leq T_l. \tag{1.5}$$

The fund's distribution rate ν_t is assumed to follow a stochastic process $\{\nu_t, 0 \leq t \leq T_l\}$ with specification:

$$\nu_t = \nu t + \sigma_\nu B_{\nu,t}, \tag{1.6}$$

with constants $\nu, \sigma_\nu > 0$; $B_{\nu,t}$ is a fourth standard Brownian motion for which it is assumed that $dB_{\nu,t} dB_{M,t} = \rho_\nu dt$ and $dB_{\nu,t} dB_{\epsilon,t} = 0$.

The ordinary differential equation for $t < T_l$ in (1.5) illustrates that capital repayments occur at some rate ν_t from the investment value. This essentially means assuming a dividend process for private equity funds during their lifetimes, similar to the specification used by Takahashi and Alexander (2002) and Malherbe (2004). Moreover, the assumption that the distribution rate dynamics can be described by the Lévy process given in (3.5) is plausible because this specification reflects well the typical lifecycle of private equity fund investments where (average) capital distributions are low at the beginning and increase over the bounded life of a fund, as more and more investments of the fund are being gradually exited. This behavior can easily be inferred from the unconditional expectation of (3.5), $E[\nu_t] = \nu t$, which increases linearly over time.

In addition, by allowing for a non-zero correlation $|\rho_\nu| \leq 1$ between changes in the distribution rate and aggregate stock market returns, I incorporate the important possibility that the overall state of the economy (or stock market) affects the speed of capital distributions. For instance, if $1 \geq \rho_\nu > 0$, then fast capital distributions will become more likely during stock market booms.

Finally, the integral specification for R_t in (1.5) takes into account that a private equity fund is fully liquidated at the end of its legal lifetime T_l. Therefore, cumulated capital distributions over the entire life of a fund must also include the final liquidation proceeds of the assets of the fund at maturity T_l, i.e., $R_{T_l} = \int_0^{T_l} \nu_u V_u du + V_{T_l}$.

1.2 Investor's Position

To define the dynamics of an investor's position when investing in a private equity fund, assume that undrawn capital commitments, $C_0 - D_t$, remain invested at some constant (riskless) rate r_c until they are actually drawn by the fund management, and that intermediate capital distributions are reinvested at the rate r_c for the remaining lifetime of the fund. As defined above, let V_t be the net-asset-value of a private equity fund investment at time t for a given initial commitment C_0. Additionally, denote by

C_t the investor's cash position at time t. Then, the value P_t of the investor's position at time t is given by:

$$P_t \equiv V_t + C_t, \tag{1.7}$$

which reflects that the investor's position is made of the net-asset-value of the private equity fund investment and of his cash holdings. The dynamics of V_t and C_t are given by:

$$dV_t = V_t(\mu_V dt + \beta_V \sigma_M dB_{M,t} + \sigma_\epsilon dB_{\epsilon,t}) + dD_t - dR_t, \tag{1.8}$$

$$dC_t = C_t r_c dt - dD_t + dR_t, \tag{1.9}$$

with $V_0 = 0$ and $C_0 > 0$.

Equations (1.8) and (1.9) show that capital drawdowns (dD_t) decrease the value C_t, as the investor has to reduce his cash holdings to meet capital calls, whereas they increase the net-asset-value V_t of the fund. In contrast, capital distributions (dR_t) of a private equity fund decrease the net-asset-value V_t and increase the value of C_t, as they are assumed to be reinvested at the constant rate r_c.

From Itô's Lemma and equations (1.7-1.9), it follows that value dynamics of the investor's position are given by

$$\begin{aligned} dP_t &= dV_t + dC_t \\ &= V_t(\mu_V dt + \beta_V \sigma_M dB_{M,t} + \sigma_\epsilon dB_{\epsilon,t}) + C_t r_c dt, \end{aligned} \tag{1.10}$$

which can also be restated in term of the instantaneous returns dP_t/P_t. This yields

$$\frac{dP_t}{P_t} = w_{V,t}(\mu_V dt + \beta_V \sigma_M dB_{M,t} + \sigma_\epsilon dB_{\epsilon,t}) + (1 - w_{V,t}) r_c dt, \tag{1.11}$$

with $w_{V,t} = V_t/P_t$ being the fraction of the investor's position invested in the private equity fund at time t and $(1 - w_{V,t}) = C_t/P_t$ being the fraction invested riskless at time t.

Note that equation (1.11) implies that the distribution of returns is non-stationary and cannot be described by a normal distribution because the fraction $w_{V,t}$ follows a (non-stationary) stochastic process over time. Thus, standard techniques to calculate the *VaR* that are based on the assumption of normally distributed and stationary returns cannot be applied in the model.

2 Risk Measures

Given the special features of private equity fund investments, a risk management framework has to capture several sources of risk. This section reviews the main sources of risk and develops adequate measures to capture them.

2.1 Sources of Risk

Private equity fund investors are exposed to three main sources of risk:[5]

- *Market Risk*: The risk of losses in the market prices of the portfolio companies held by a fund exposes investors to market risk.
- *Liquidity Risk*: The illiquidity of private equity partnership interests exposes investors to asset liquidity risk associated with selling in the secondary markets at a discount on the fund's net-asset-value.
- *Funding (or Cash Flow) Risk*: The unpredictable timing of cash flows poses funding and cash flow risks to investors. In particular, capital commitments are contractually binding and defaulting on these payments can result in the loss of the entire private equity partnership interest.

[5] Besides, private equity fund investors are also exposed to several other sources of risk, which are not considered in more detail here. These include (but not limited to) the risk of selecting a low quality fund manager, interest rate risk, and foreign exchange rate risk.

Accounting for these main risk sources, this section develops three risk measures for private equity fund investments: Value-at-Risk (*VaR*), Cash-Flow-at-Risk (*CFaR*), and Liquidity-Adjusted Value-at-Risk (*LVaR*).

2.2 Value-at-Risk (*VaR*)

Currently the most widely used measure of exposure to market risk is Value-at-Risk, usually abbreviated to *VaR*. *VaR* was developed and adopted in response to financial disasters, such as those at Baring's Bank, Orange County, and Metallgesellschaft. After JP Morgan has developed the measure in the early 1990s, the concept eventually became accepted by the general finance community when the Basel Committee on Banking Supervision allowed financial institutions to quantify their market risks with internal *VaR* models. A standard treatment of *VaR* can be found in Jorion (2001). Today, the financial industry has standardized on the following definition of *VaR*:

> $VaR(\alpha)$: *It is the loss of a financial position or portfolio, which is exceeded with some given probability α, over a given time horizon.* (see, for example, Jorion (2001))

Assuming a time horizon that equals h, this definition translates into the *VaR* of the investor's position under the developed model setting as

$$Prob(P_t - P_{t+h} \leq VaR_{t,h}(\alpha)) = 1 - \alpha, \qquad \alpha \in [0, 1], \tag{2.1}$$

where P_t is the value of the investor's position with dynamics given by (1.10). Intuitively, this *VaR* can be interpreted as the worst loss that occurs for the private equity fund investor over a given time interval h, under "normal market conditions".

Note that the time index t in $VaR_{t,h}(\alpha)$ accounts for the fact that stepwise capital drawdowns and intermediate capital distributions lead to a situation where the model *VaR* is not time-invariant. Because the fractions of the investor's position invested into the private equity fund and into the riskless asset change stochastically, $VaR_{t,h}(\alpha)$ also changes over the lifetime of the fund. In contrast, standard *VaR* calculations typically assume a static setting where the relative weights of all positions are not changed over time.

2.3 Liquidity-Adjusted Value-at-Risk (*LVaR*)

The *VaR* measure defined in (2.1) is based on the implicit assumption that an investor can sell his position in the fund at any time $t \in [0, T_l]$ at the fund's current net-asset-value V_t. In reality, stakes in private equity funds are highly illiquid in the sense there is no organized and liquid market where funds can be traded at low costs. Investors that have private equity exposure and need immediate liquidity must sell their interests in a private equity fund on the secondary private equity markets. However, these secondary markets are still relatively immature and pricing is highly inefficient. Consequently, selling fund positions is typically only possible at some discount to the current net-asset-value (see Kleymenova et al. (2012)). Figure 1 illustrates quarterly median secondary market discounts for the buyout and venture capital segment that can be obtained from the Preqin Secondary Market Monitor for the period ranging from March 2003 to March 2013. The figure shows that the range of discounts is large and depends on the specific type of asset and market environment. Thus, investors can be exposed to a large liquidity risk when they have to sell a private equity fund prior to its liquidation. Ignoring this form of liquidity risk can substantially understate the investor's true risk exposure.

In order to account for this effect, the *VaR* calculation presented above can be extended to a Liquidity-Adjusted Value-at-Risk, abbreviated to *LVaR* in the following. The main idea of this *LVaR* is to include secondary markets discounts as an exogenous liquidity risk in the *VaR* calculation. This is done by taking into account that the liquidation value of a private equity fund is typically not equal to its current net-asset-value, but has to be adjusted for secondary market discounts. Stated differently, while the standard *VaR* measure presented above considers the worst net-asset-value of a fund for some confidence level α, the *LVaR* considers the worst transaction price that could be obtained for a fund on the secondary private equity markets for some confidence level α. This idea translates into the following formal definition of $LVaR(\alpha)$:

$$Prob(P_t - [(1 - \pi_{t+h})V_{t+h} + C_{t+h}] \leq LVaR_{t,h}(\alpha)) = 1 - \alpha, \qquad \alpha \in [0, 1], \tag{2.2}$$

where π_{t+h} is the secondary market discount for the fund at time $t + h$. Because the investor's cash position C_{t+h} is not affected by secondary market discounts, the term $[(1 - \pi_{t+h})V_{t+h} + C_{t+h}]$ equals the value of the investor's total position after liquidation of the fund on the secondary markets.

The *LVaR* defined in (2.5) requires an assumption on the dynamics of the secondary market discounts. As secondary market discounts typically vary over time with the prevailing market conditions (see Figure 1), an exogenous stochastic process is introduced for the discount rate π_t. To incorporate that discounts can also become negative, an Ornstein-Uhlenbeck process is employed.

Assumption 2.1 *The secondary market discount rate π_t is assumed to follow a stochastic process $\{\pi_t, 0 \leq t \leq T_l\}$, with specification given by*

$$d\pi_t = \kappa_\pi(\theta_\pi - \pi_t)dt + \sigma_\pi dB_{\pi,t}, \tag{2.3}$$

where $\theta_\pi > 0$ is the long-run mean of the discount rate, $\kappa_\pi > 0$ is the rate of reversion to this mean, and $\sigma_\pi > 0$ reflects the volatility of the discount rate; $B_{\pi,t}$ is a fifth standard Brownian motion for which it is assumed that $dB_{\pi,t}dB_{M,t} = \rho_\pi dt$ and $dB_{\pi,t}dB_{\epsilon,t} = 0$.

This process displays mean-reversion property, i.e., the discount rate randomly fluctuates around the (long-run) mean level θ_π. Additionally, the specification also allows for a non-zero correlation ρ_π between changes in secondary market discounts and aggregate stock market returns. This is done here to take into account that the overall state of the economy (or stock market) may also affect discounts on the secondary markets.

2.4 Cash-Flow-at-Risk (*CFaR*)

Since private equity fund investors are in general also concerned with the size and timing of the fund cash flows (i.e., capital drawdowns and distributions), a third adequate risk measure that can be defined for private equity fund investments is Cash-Flow-at-Risk, abbreviated to *CFaR* in the following. The measure $CFaR(\alpha)$ is defined here as the change (loss) in the investor's cash position, which is exceeded with some given probability α, over a given time horizon.

Formally, assuming a time horizon that equals h, this definition of *CFaR* translates into

$$Prob(C_t - C_{t+h} \leq CFaR_{t,h}(\alpha)) = 1 - \alpha, \qquad \alpha \in [0, 1], \tag{2.4}$$

where C_t is the investor's cash position with dynamics given by (1.9).

The interpretation of this measure changes with the stage in the lifecycle of the fund. In the early times of a fund's life, when the funds starts to stepwise draw down capital from the investors and builds up the investment portfolio, $CFaR_{t,h}(\alpha)$ is positive and gives the maximum amount of cash that the investor must hold in reserve over the time horizon h in order to being able to meet capital calls of the fund management. In the later times of a fund's life, when the funds starts to gradually exit its investments and distributes the proceeds to the investors, $CFaR_{t,h}(\alpha)$ will eventually turn negative and its absolute value gives an indication of the minimum amount of cash distributed by the private equity fund over the time horizon h.

Note that the *VaR* defined in (2.1) and *CFaR* defined in (2.4) are equal when calculated over the total lifetime T_l of the fund, i.e.,

$$VaR_{0,T_l}(\alpha) = CFaR_{0,T_l}(\alpha). \tag{2.5}$$

This relationship follows because the change in value of the investor's position over the total fund lifetime, $P_0 - P_{T_l}$, equals the change in his cash position, $C_0 - C_{T_l}$, as funds are fully liquidated at the end of their lifetime T_l, and consequently $V_0 - V_{T_l} = 0$ holds per definition.

3 Monte-Carlo Simulation

The model developed in the previous sections is path dependent: The net-asset-value V_t of a fund as well as the investor's cash position C_t at any time t depend on the full history of past capital drawdowns and distributions. These path dependencies can easily be taken into account by using Monte Carlo simulations to calculate the developed risk measures of a private equity fund investment. This section develops a discrete-time version of the model and shows how risk measures can be estimated using Monte-Carlo simulations.

3.1 Discrete-Time Version of the Model

In order to implement the Monte Carlo simulation, the time interval $[0, T_l]$ is divided into K discrete intervals, each of length Δt. Then, all relevant quantities are considered at equidistant time points, $t_k = k\Delta t$, where $k = 1, \ldots, K$ and $K = T_l/\Delta t$.

An appropriate scheme for discretizing the value dynamics of a private equity fund is the *Euler scheme*.[6] Applying this scheme to SDE (1.1) gives

$$\begin{aligned} V_{k+1} = & V_k[1 + \mu_V \Delta t + \beta_M \sigma_M \epsilon_{M,k+1} \sqrt{\Delta t} + \sigma_\epsilon \epsilon_{\epsilon,k+1} \sqrt{\Delta t}] \\ & - \Delta R_{k+1} + \Delta D_{k+1}, \end{aligned} \tag{3.1}$$

where $\epsilon_{M,1}, \epsilon_{M,2}, \ldots, \epsilon_{M,K}$ and $\epsilon_{\epsilon,1}, \epsilon_{\epsilon,2}, \ldots, \epsilon_{\epsilon,K}$ are *i.i.d.* sequences of standard normal

[6]For an arbitrary SDE, $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$, the Euler scheme takes the form

$$X_{k+1} = X_k + \mu(X_k)\Delta t + \sigma(X_k)\sqrt{\Delta t}\ \epsilon_{k+1},$$

where ϵ_{k+1} is a standard normal variable. For more details on the approximation of SDEs in discrete-time, see Glasserman (2003) or Kloeden and Platen (1999).

variables that are uncorrelated to each other.

The dynamics of the capital drawdowns, equation (1.3), can be represented in discrete-time as[7]

$$\Delta D_{k+1} = \delta_{k+1}(C_0 - D_k)\Delta t, \tag{3.2}$$

with the drawdown rate dynamics given by

$$\delta_{k+1} = \delta + \sigma_\delta\ \epsilon_{\delta,k+1}\sqrt{(k+1)\Delta t}. \tag{3.3}$$

where $\epsilon_{\delta,1}, \epsilon_{\delta,2}, \dots, \epsilon_{\delta,K}$ is a third *i.i.d.* sequence of standard normal variables.

Similarly, an appropriate discrete-time version of the dynamics of the capital distributions, specification (1.5), is

$$\Delta R_{k+1} = \nu_{k+1} V_k \Delta t, \quad \text{if} \quad k+1 < K, \quad \text{and} \quad \Delta R_K = V_K, \tag{3.4}$$

with distribution rate dynamics given by

$$\nu_{k+1} = \nu(k+1)\Delta t + \sigma_\nu \epsilon_{\nu,k+1}\sqrt{(k+1)\Delta t}, \tag{3.5}$$

where $\epsilon_{\nu,1}, \epsilon_{\nu,2}, \dots, \epsilon_{\nu,K}$ is a fourth *i.i.d.* sequence of standard normal variables.

The constant correlation ρ_δ of the drawdown rate with the market returns and the constant correlation ρ_ν of the distribution rate with the market returns can be achieved by setting

$$\epsilon_{\delta,k} = \rho_\delta \epsilon_{M,k} + \sqrt{1-\rho_\delta^2}\ \epsilon_{1,k}, \tag{3.6}$$

$$\epsilon_{\nu,k} = \rho_\nu \epsilon_{M,k} + \sqrt{1-\rho_\nu^2}\ \epsilon_{2,k}, \tag{3.7}$$

where $\epsilon_{1,k}$ and $\epsilon_{2,k}$ are standard normal variables that are uncorrelated with each other

[7] For simplicity, it is assumed here that the commitment period T_c equals the legal lifetime of the fund T_l, i.e., $T_c = T_l$.

(and with all other random variables introduced here).

Using the Euler scheme to approximate the discount rate process, SDE (2.3), discrete-time dynamics are given by

$$\pi_{k+1} = \pi_k + \kappa_\pi(\theta_\pi - \pi_k)\Delta t + \sigma_\pi\ \epsilon_{\pi,k+1}\sqrt{\Delta t}, \tag{3.8}$$

where $\epsilon_{\nu,1}, \epsilon_{\nu,2}, \ldots, \epsilon_{\nu,K}$ is *i.i.d.* is another sequence of standard normal variables that has a constant correlation ρ_π with the sequence $\epsilon_{M,1}, \epsilon_{M,2}, \ldots, \epsilon_{M,K}$, i.e.,

$$\epsilon_{\nu,k} = \rho_\nu \epsilon_{M,k} + \sqrt{1-\rho_\nu^2}\ \epsilon_{3,k}, \tag{3.9}$$

where $\epsilon_{3,k}$ is a standard normal variable that is uncorrelated with all other random variables introduced before.

Using these specifications, the discrete-time dynamics of the investor's position and of his cash holdings can be represented as

$$P_{k+1} = P_k + V_k(\mu_V \Delta t + \beta_M \sigma_M \epsilon_{M,k+1}\sqrt{\Delta t} + \sigma_\epsilon \epsilon_{\epsilon,k+1}\sqrt{\Delta t}) + C_k r_f \Delta t, \tag{3.10}$$

$$C_{k+1} = C_k(1 + r_c)\Delta t - \Delta D_t + \Delta R_t. \tag{3.11}$$

3.2 Estimating Risk Measures by Simulation

To numerically evaluate the risk measures developed in Section 2, consider a Monte Carlo sampling experiment composed of M independent replications of the discrete-time approximations of the model given above. To derive an estimate of the standard *VaR* defined in (2.1), denote by $L_{k,i}$ the kth observation of the investor's loss over the time horizon h in the ith replication,

$$L_{k,i} = P_{k,i} - P_{k+h,i}. \tag{3.12}$$

Let $\hat{F}_{L,k}(x)$ denote the approximated distribution of losses L based on the M simulated replications,

$$\hat{F}_{L,k}(x) = \frac{1}{M}\sum_{i=1}^{M} 1_{\{L_{k,i} \leq x\}}. \tag{3.13}$$

Using the approximated distribution $\hat{F}_{L,k}(x)$, a simple estimate of *VaR* at the confidence level α is given by the empirical quantile, i.e.,

$$\widehat{VaR}_{k,h}(\alpha) = \hat{F}^{-1}_{L,k}(1-\alpha), \tag{3.14}$$

with the inverse of the piecewise constant function $\hat{F}_{L,k}$ defined by $\hat{F}^{-1}_{L,k}(u) = \inf\{x : \hat{F}_{L,k}(x) \geq u\}$. Under minimal conditions (see, for example, Serfling (1980)), the estimated $\widehat{VaR}_{k,h}(\alpha)$ converges to the true model $VaR_{k,h}(\alpha)$ with probability 1, as $M \to \infty$.

Similarly, the measures *LVaR* and *CFaR* defined in (2.4) and (2.5) can be estimated by

$$\widehat{LVaR}_{k,h}(\alpha) = \hat{F}^{-1}_{L^{ad},k}(1-\alpha), \tag{3.15}$$

$$\widehat{CFaR}_{k,h}(\alpha) = \hat{F}^{-1}_{L^{c},k}(1-\alpha), \tag{3.16}$$

where $\hat{F}_{L^{ad},k}(x)$ denotes the approximated distribution of liquidity adjusted losses L^{ad} in the investor's position, with $L^{ad}_{k,i} = P_{k,i} - [(1-\pi_{k+h,i})V_{k+h,i} + C_{k+h,i}]$, and $\hat{F}_{L^c,k}(x)$ denotes the approximated distribution of losses L^c in the investor's cash position, with $L^c_{k,i} = C_{k,i} - C_{k+h,i}$.

4 Numerical Analysis

This section illustrates the developed model and risk measures through a numerical example and discusses its implications. The model analysis illustrates how the risk measures evolve over the lifecycle of a fund and performs a sensitivity analysis that highlights the effects of shocks in the main model parameters.

4.1 Calibrated Model Parameters

The analysis starts with the question of what are reasonable parameter values for the model. Where possible, baseline parameters are chosen in the following such that they correspond to an investment in an average buyout fund.

A. Risk and Return Characteristics

To calibrate the risk and return characteristics of buyout funds, I mainly rely on estimated parameters from Metrick and Yasuda (2010) and Ang et al. (2013). I use a market beta of a private equity fund of $\beta_V = 1.3$ and an alpha of $\alpha_V = 0.04$, which is consistent with recent estimation evidence from Ang et al. (2013), who find average CAPM betas of buyout funds that equal 1.31 and average CAPM alphas that equal 4% per annum.

Metrick and Yasuda (2010) find an annual volatility of $\sigma_i = 60\%$ per individual buyout investment and an average pairwise correlation of $\rho_{ii} = 0.2$ between any two investments. They report that the average buyout fund invests in around 15 companies (with a median of 12). The average holding period of a buyout investment is around four years (see Franzoni et al. (2012)). With a typical fund lifetime of 12 years, the average number of investments running at any time during a fund's lifetime is $N = (15 \times 4)/12 = 5$. Using these values, the total variance of fund returns, σ_V^2, can be approximated by

$$\sigma_V^2 = \frac{1}{N}\sigma_i^2(1 - \rho_{ii}) + \sigma_i^2\rho_{ii}, \tag{4.1}$$

which yields an annual volatility of 0.36, which is rounded to $\sigma_V = 40\%$.

Like Metrick and Yasuda (2010), an annual risk-free rate of $r_f = 5\%$ is used. For the stock market, an annual volatility of $\sigma_M = 15\%$ with an expected return of $\mu_M = 11\%$ is assumed, which implies a risk premium of $\mu_M - r_f = 6\%$. Given this level of stock market volatility, the annual idiosyncratic volatility of a buyout fund is $\sigma_\epsilon = \sqrt{\sigma_V^2 - \beta_V^2 \sigma_M^2} = 35\%$, and the correlation between private equity fund returns and aggregate stock market returns yields a reasonable $\rho_V = \beta_V \sigma_M / \sigma_V = 0.49$.

B. Drawdown and Distribution Dynamics

To calibrate the drawdown and distribution rate dynamics, I use cash flow data at the individual fund level, which is net of management fees and carried interest payments. The data has been provided by the Center of Private Equity Research (Cepres), which maintains one of the largest databases of precisely timed deal and fund level cash flows.[8] To minimize a possible impact by estimated net asset values of unrealized investments, only funds with vintage years from prior to 2001 are being used. This gives a comprehensive sample of monthly cash flows of 200 buyout funds.

To estimate the model parameters, I employ the concept of *conditional least squares* (CLS). Appendix A shows that this method leads to the following least squares estimators for the drawdown rate parameters δ and σ_δ^2:

$$\hat{\delta} = \frac{1}{\Delta t} \frac{\sum_{k=1}^{K} (\bar{D}_k - \bar{D}_{k-1})(1 - \bar{D}_{k-1})}{\sum_{k=1}^{K} (1 - \bar{D}_{k-1})^2}, \tag{4.2}$$

$$\hat{\sigma}_\delta^2 = \frac{1}{K} \sum_{k=1}^{K} \frac{[\bar{D}_k - \bar{D}_{k-1} - \hat{\delta}(1 - \bar{D}_{k-1})\Delta t]^2}{(1 - \bar{D}_{k-1})^2 (\Delta t)^3 k}, \tag{4.3}$$

where $\bar{D}_k$ denotes average cumulated capital drawdowns of the N sample funds at time k.

The corresponding least squares estimators for the distribution rate parameters ν

[8] Earlier versions of this dataset have, for example, been used by Krohmer et al. (2009), Cumming et al. (2010), Franzoni et al. (2012), and Buchner and Stucke (2014).

and σ_ν^2 are given by:

$$\hat{\nu} = \frac{1}{\Delta t} \frac{\sum\limits_{k=1}^{K} (\bar{R}_k - \bar{R}_{k-1}) k \bar{V}_{k-1}}{\sum\limits_{k=1}^{K} k^2 \bar{V}_{k-1}^2}, \tag{4.4}$$

$$\hat{\sigma}_\nu^2 = \frac{1}{K} \sum_{k=1}^{K} \frac{[\bar{R}_k - \bar{R}_{k-1} - \hat{\nu} k \bar{V}_{k-1} (\Delta t)^2]^2}{(\bar{V}_{k-1})^2 (\Delta t)^3 k}, \tag{4.5}$$

where $\bar{R}_k$ denotes average cumulated capital distributions of the N sample funds at time k and $\bar{V}_k$ denotes average net-asset-values of the N sample funds at time k. The average net-asset-values of the funds, $\bar{V}_k$, cannot directly be observed from the underlying cash flow data. I approximate them by the following relationship

$$\bar{V}_k = \sum_{i=1}^{k} (\overline{\Delta D}_i - \overline{\Delta R}_i)(1 + \overline{IRR})^{k-i}, \tag{4.6}$$

where $\overline{IRR}$ is the mean internal rate of return of the sample funds, $\overline{\Delta D}_i = (\bar{R}_i - \bar{R}_{i-1})$ and $\overline{\Delta R}_i = (\bar{R}_i - \bar{R}_{i-1})$.

Applying this estimation methodology yields the following set of parameters: $\hat{\delta} = 0.41$, $\hat{\sigma}_\delta = 0.21$, $\hat{\nu} = 0.08$, and $\hat{\sigma}_\nu = 0.11$.

Simulating the fund cash flows also requires assumptions on the correlations of the drawdown and distribution rate with aggregate market returns. Empirical evidence by Robinson and Sensoy (2011) shows that capital calls and distributions have a systematic component that is pro-cyclical and that distributions are more sensitive to market conditions than calls. In line with these observation, I assume that both correlations are positive and that $\rho_\nu > \rho_\delta$. The specific correlations used in the baseline specification are $\rho_\delta = 0.5$ and $\rho_\nu = 0.8$.

C. Secondary Market Discount Dynamics

To calibrate the secondary market discount rate process, SDE (2.3), I use the quarterly data on median secondary market discounts for the buyout segment from the Preqin Secondary Market Monitor. The data is illustrated in Figure 1.

Using this data, model parameters κ_π, θ_π, and σ_π are estimated by applying a maximum likelihood (ML) method on a discrete-time counterpart of the Ornstein-Uhlenbeck process (2.3).[9] This approach yields the following parameters for the buyout segment: $\kappa_\pi = 0.42$, $\theta_\pi = 0.16$, and $\sigma_\pi = 0.16$. This set of parameters suggests that the long-run mean of the discount rate equals 16% and that the half-life of the mean-reversion, i.e., the average time it takes the process to get pulled half way back to its long-run mean, equals $\ln 2/\kappa_\pi = 1.65$ years. In addition, in the baseline case, the initial discount rate π_0 is set to the long-run mean θ_π.

Coefficient ρ_π is estimated by the correlation between quarterly changes in median secondary market discounts and the corresponding quarterly S&P 500 returns. This yields $\rho_\pi = -0.60$, implying that the overall market conditions affect the pricing in the secondary markets and that discounts increase during stock market downturns.

Table 1 provides a summary of the baseline parameter values used for the subsequent numerical analysis.

4.2 Fund Dynamics and Risk Measures

The analysis now turns to the numerical results. All subsequent results are generated by using Monte-Carlo simulations with 500.000 iterations.

A. Fund Dynamics

To illustrate the model fund dynamics, in addition to the baseline parameters given above, assume that the fund has a typical legal lifetime of 12 years and that the investor's initial capital commitment equals $C_0 = 100$. Using this set of parameters, Figure 2 illustrates the dynamics of the cumulated capital drawdowns, capital distributions, net fund cash flows, and of the fund value.

[9] See Gourieroux and Jasiak (2001), p. 290, for the resulting closed-form estimators.

It is important to acknowledge that the basic patterns of the model graphs shown in Figure 2 conform to reasonable expectations of private equity fund behavior. In particular, the cash flow streams that the model generates naturally exhibit a lag between the capital drawdowns and distributions, thus reproducing the typical development cycle of a fund and leading to the private equity characteristic J-shaped curve for the cumulated net fund cash flows that can be observed on the left of Figure 2 (b). In addition, note that the value dynamics of the fund shown on the right of Figure 2 (b) are also in line with expectations. Specifically, the value of the fund increases over time as the investment portfolio is build up and decreases towards the end as fewer and fewer investments are left to be harvested. The model also captures the erratic nature of real world private equity fund cash flows and value dynamics, which is indicated in Figure 2 by the 80% confidence intervals shown. Furthermore, it is important to stress that the model is flexible enough to generate the potentially many different patterns of capital drawdowns and distributions. By altering the main model parameters, both timing and magnitude of the fund cash flows can be controlled.

B. Value-at-Risk Dynamics

Figure 3 illustrates the simulated *VaR* dynamics. Figure 3 (a) shows how the *VaR* at fund initiation (i.e., at $t = 0$), $VaR_{0,h}(\alpha)$, changes with the time horizon h. The confidence levels used are 10%, 5%, and 1%. Figure 3 (a) reveals interesting patterns. For any given confidence level, as the time horizon h increases, $VaR_{0,h}(\alpha)$ initially rises sharply but then peaks and turns slightly down. The initial sharp increase is in line with expectations. As the fund gradually draws down capital and builds up the investment portfolio, the investor's risk exposure and consequently also his maximum possible loss that can occur at some given confidence level both increase markedly. The slight decrease in *VaR* for long horizons h is due to compounding effects in fund returns, which reduce the probability of large losses over long time horizons. The reduction in *VaR* is small however, because intermediate capital distributions limit compounding effects in fund returns. The behavior of the *VaR* also depends on the chosen confidence level: for high confidence levels, $VaR_{0,h}(\alpha)$ peaks more quickly and then falls more pronounced; for low confidence levels, $VaR_{0,h}(\alpha)$ peaks more slowly and stays near its maximum level afterwards. Note that the maximum value of $VaR_{0,h}(\alpha)$ is well below the investor's initial capital commitment of 100 for all confidence levels considered. Thus, the maximum possible loss of a private equity fund investment is typically much lower than the investor's total capital commitment.

Figure 3 (b) further illustrates the *VaR* dynamics over the fund lifetime for a fixed time horizon of three months, i.e., $h = 0.25$. As expected, the fixed horizon *VaR* of the fund investment, $VaR_{t,0.25}(\alpha)$, increases as the fund draws down capital and peaks around the maximum average fund value (see also Figure 2). After the maximum level has been reached, $VaR_{t,0.25}(\alpha)$ decays rapidly towards zero as capital distributions of the fund stepwise decrease the investor's risk exposure. This basic pattern can be observed for all confidence levels considered. However, the results also show here that $VaR_{t,0.25}(\alpha)$ peaks more quickly for lower confidence levels α.

Overall, the dynamics illustrated in Figure 3 underline that there exists private equity specific patterns in *VaR* dynamics that a sensible risk management framework must take into account. These specific patterns are a direct result of the complex cash flow structures of private equity fund investments, which lead to a situation where the mix of capital invested in the fund and cash holdings changes dynamically over the fund lifetime. In contrast, standard *VaR* implementations typically assume a static setting where the mix in a position is unchanged over the time horizon of the *VaR*. Thus, when applied to private equity fund investments, standard *VaR* calculations can lead to biased conclusions about the investor's true risk exposure.

C. Liquidity-Adjusted Value-at-Risk Dynamics

Figure 4 compares the standard *VaR* to the liquidity-adjusted *LVaR* that is obtained by correcting fund values for potential secondary market discounts. The confidence level used for this comparison is 1%. As expected, the results highlight that illiquidity increases the investor's risk exposure. In this sense, Figure 4 (a) shows that the $LVaR$ at fund initiation, $LVaR_{0,h}(\alpha)$, is typically higher than the standard $VaR_{0,h}(\alpha)$. However, the figure also adds that illiquidity only increases the investor's risk exposure if the time horizon h under consideration is smaller than the funds's total legal lifetime T_l, as $LVaR_{0,T_l}(\alpha) = VaR_{0,T_l}(\alpha)$. The economic rationale behind this result is straightforward. Because a private equity fund is fully liquidated at the end of its legal lifetime T_l, liquidity risk is fully resolved at T_l. Thus, liquidity risk is only of importance if there is a mismatch between the investor's expected holding horizon h and a fund's legal lifetime T_l. For long-term investor's with sufficient cash reserves, liquidity risk is consequently only of minor importance.

The dynamics of the three months *LVaR*, $LVaR_{t,0.25}(\alpha)$, displayed in Figure 4 (b) further highlight that the effects of illiquidity on the investor's risk exposure can be substantial over the life of the fund. For the given confidence level of 1%, the investor's maximum loss that can occur over a period of three months increases from around 41 to 66. Thus, for investors who face the risk of surprise liquidity shocks and need to sell their private equity positions on the secondary markets, ignoring liquidity risk can lead to substantially underestimated risk exposures.

D. Cash-Flow-at-Risk Dynamics

The dynamics of the developed *CFaR* measure are illustrated in Figure 5. The results highlight that an investor is exposed to substantial funding (or cash flow) risk when investing into private equity funds. Similar to above, Figure 5 (a) shows how the *CFaR* at fund initiation, $CFaR_{0,h}(\alpha)$, changes with the time horizon h. In line with expectations, $CFaR_{0,h}(\alpha)$ increases quickly during the commitment period when the fund starts to draw down capital and reaches a maximum after around 12 to 15 quarters, depending on the chosen confidence level α. It is important to note that the maximum level is below the investor's initial capital commitment, $C_0 = 100$, for all confidence levels considered. The reason for this is that private equity funds typically start to distribute capital before the committed capital has actually been fully drawn. Thus, investors typically do not need to reserve all committed capital in cash as distributions from existing investments can be used to partially finance open commitments. After the maximum level is attained, $CFaR_{0,h}(\alpha)$ decreases because capital distributions of the fund gradually increase the investor's cash position. Besides, it is also interesting to see how the behavior of $CFaR_{0,h}(\alpha)$ changes with the chosen confidence level α. For short time horizons h, differences between the curves for different levels of α are small but increase as h tends towards the fund lifetime T_l. The rationale behind this result is that capital distributions involve higher levels of uncertainty than capital drawdowns (see also Figure 1 (a)).

The fixed horizon $CFaR$ for $h = 0.25$ shown in Figure 5 (b) further illustrates how cash flow risk changes with the development stage of the fund. In the early stages of a fund's life, when the fund starts to call capital from the investors, $CFaR_{t,0.25}(\alpha)$ is large and gives a measure of funding risk, i.e., the maximum amount of cash that the investor must hold in reserve over next quarter in order to being able to meet capital calls. In the later times of a fund's life, when the funds starts to gradually exit its portfolio companies and returns the proceeds to the investors, $CFaR_{t,h}(\alpha)$ decreases and gives a measure of distribution risk, i.e., the negative value of $CFaR_{t,h}(\alpha)$ gives an indication of the minimum amount of cash distributed by the private equity fund over the next quarter.

E. Sensitivity Analysis

In the environment of high uncertainty that characterizes private equity, there are limits to the predictive value of model parameters that are calibrated from historical data. It is therefore meaningful to evaluate and quantify the impact of parameter shocks that could materially change risk projections. Figures 6 to 8 illustrate a number of such shocks that should be applied for stress testing.

Figure 6 identifies the main drivers of VaR. The key risk parameters in the model are the market beta of a fund, β_V, and its idiosyncratic volatility, σ_ϵ. In line with expectations, the figure shows that a increase in these two risk parameters, proxied here by multiplying both parameters by 1.5, leads to a substantial increase in the fixed horizon

VaR. Moreover, the VaR dynamics also depend on the drawdown and distribution rate parameters. The figure shows that a higher mean drawdown rate δ results in a faster investment pace and therefore leads to a faster increase in *VaR* in the early stages of a fund's life and higher maximum *VaR* levels. The effect of a decrease in the mean distribution rate ν, which may occur during an economic downturn when exit opportunities for private equity funds deteriorate, is to increase *VaR*, particularly in the later phases of a fund's life. This holds because a lower parameter ν leads to a higher average investment duration, which increases the investor's risk exposure.

In addition to these model parameters, the *LVaR* measure is also driven by dynamics of the secondary market discounts. Figure 7 shows that a higher long-run mean discount rate θ_π and a higher volatility σ_π increase the fixed horizon *LVaR*. The effect of a higher mean reversion rate κ_π is to slightly decrease *LVaR*, which holds because a high mean reversion rate partly dampens volatility. Additionally, increasing the correlation ρ_π of the discount rate with stock market returns increases *LVaR*, because this leads to a situation where high fund values coincide with high discount rates.

Finally, Figure 8 highlights the main drivers of the developed *CFaR* measure. The results indicate that a higher volatility σ_δ of the drawdown rate increases funding risk at the start of a fund, whereas a higher volatility of the distribution rate increases the uncertainty about cash flows in the later stages of a fund's life.

5 Conclusion

Although risk management has been a well-ploughed field in financial modeling for over three decades, the understanding how to correctly quantify and manage the risks of investing in private equity remains limited. Because of institutional features that complicate the use of standard risk management tools, the majority of investors today still employ relatively simplistic approaches to measuring and reporting the risks of investing in private equity. However, with growing exposure to private equity, it has become more important to fully understand and correctly quantify the risks of investing in this asset class. Moreover, more sophisticated risk management will also be an essential catalyst for further growth of the asset class. Surprisingly, the academic literature has largely overlooked this fact; at present, we lack an adequate risk model for private equity that allows accurately addressing all relevant risks.

The objective of this paper is to fill this gap by developing the first comprehensive risk management framework for private equity fund investments. The framework addresses the three main sources of risks that private equity investors face when investing in this asset class: *market risk*, *liquidity risk*, and *cash flow risk*. Underlying the framework is a stochastic model for the value and cash flow dynamics of private equity funds, which allows to derive three dynamic risk measures for private equity fund investments: Value-at-Risk (*VaR*), Liquidity-Adjusted Value-at-Risk (*LVaR*), and Cash-Flow-at-Risk (*CFaR*). The paper calibrates the model to data of buyout funds and illustrates the dynamics of the developed risk measures using Monte-Carlo simulations. The results highlight that there exists private equity specific patterns in risk dynamics that a sensible risk management framework for private equity investments must take into account.

A Appendix: Drawdown and Distribution Rate Calibration

This appendix presents the estimation methodology for the parameters of the drawdown and the distribution rate dynamics.

A. Definitions and Methodology

The objective is to estimate model parameters from the observable cash flows of the sample funds at equidistant time points $t_k = k\Delta t$, where $k = 1, \ldots, K$, and $K = T/\Delta t$. To make funds of different sizes comparable, the capital drawdowns and capital distributions of all $j = 1, \ldots, N$ sample funds are first standardized on the basis of each fund's total invested capital. In the following, let $D_{k,j}$ denote the standardized cumulated capital drawdowns of fund j up to time t_k. Standardized cumulated capital distributions of fund j up to time t_k are denoted by $R_{k,j}$. Finally, $V_{k,j}$ represents the standardized value of fund j at time t_k.

To estimate the model parameters, I use the concept of *conditional least squares* (CLS). The concept of conditional least squares, which is a general approach for estimation of the parameters involved in the conditional mean function $E[X_k|\mathcal{F}_{k-1}]$ of a stochastic process, was given a thorough treatment by Klimko and Nelson (1978). The idea behind the method is to estimate parameters from discrete-time observations $\{X_k\}$ of a stochastic process, such that the sum of squares

$$\sum_{k=1}^{K} (X_k - E[X_k|\mathcal{F}_{k-1}])^2 \tag{A.1}$$

is minimized, where $\mathcal{F}_{k-1}$ is the σ-field generated by $X_1, \ldots, X_{k-1}$. This basic idea can be slightly adapted to the particular estimation problem given here. As time-series as well as cross-sectional data of the cash flows of the sample funds is available, a natural idea is to replace X_k in relation (A.1) by the sample average $\bar{X}_k$.

B. Drawdown Rate

To derive an estimator for the drawdown rate δ, let $\bar{D}_k$ denote the sample average of the cumulated capital drawdowns at time t_k, i.e., $\bar{D}_k = \frac{1}{N}\sum_{j=1}^{N} D_{k,j}$. An appropriate goal function to estimate parameter δ is then given by

$$\sum_{k=1}^{K}(\bar{D}_k - E[\bar{D}_k|\mathcal{F}_{k-1}])^2, \tag{A.2}$$

where $\mathcal{F}_{k-1}$ is the σ-field (the available information set) generated by the sequence $\bar{D}_1, \ldots, \bar{D}_{k-1}$. The conditional expectation in (A.2) can, in discrete-time, be states as:

$$E[\bar{D}_k|\mathcal{F}_{k-1}] = \bar{D}_{k-1} + \delta(1 - \bar{D}_{k-1})\Delta t. \tag{A.3}$$

Substituting (A.3) into (A.2), the goal function to be minimized turns out to be

$$\sum_{k=1}^{K}\{\bar{D}_k - \bar{D}_{k-1} - \delta(1 - \bar{D}_{k-1})\Delta t\}^2. \tag{A.4}$$

The least-squares estimator is then the solution to the equation $\sum_{k=1}^{K}(\partial/\partial\delta)\{\bar{D}_k - \bar{D}_{k-1} - \delta(1 - \bar{D}_{k-1})\Delta t\}^2 = 0$. The resulting expression for the estimator is:

$$\hat{\delta} = \frac{1}{\Delta t}\frac{\sum\limits_{k=1}^{K}(\bar{D}_k - \bar{D}_{k-1})(1 - \bar{D}_{k-1})}{\sum\limits_{k=1}^{K}(1 - \bar{D}_{k-1})^2}. \tag{A.5}$$

To estimate the volatility σ_δ, note that the conditional variance of the average capital drawdowns in the interval $(t_{k-1}, t_k]$ can be stated as

$$\begin{aligned} E[\bar{D}_k - E[\bar{D}_k|\mathcal{F}_{k-1}]|\mathcal{F}_{k-1}]^2 &= Var[\bar{\delta}_k(1 - \bar{D}_{k-1})\Delta t|\mathcal{F}_{k-1}] \\ &= [(1 - \bar{D}_{k-1})\Delta t]^2 Var[\bar{\delta}_k|\mathcal{F}_{k-1}]. \end{aligned} \tag{A.6}$$

Using the discrete-time specification of the drawdown rate given in (3.3), the conditional variance $Var[\bar{\delta}_k|\mathcal{F}_{k-1}]$ of the average drawdown rate $\bar{\delta}_k$ is given by:

$$Var[\bar{\delta}_k|\mathcal{F}_{k-1}] = \bar{\sigma}_\delta^2 k\Delta t. \tag{A.7}$$

Substituting equation (A.3) and (A.7) into (A.6), an appropriate estimator of the variance $\bar{\sigma}_\delta^2$ turns out to be

$$\hat{\bar{\sigma}}_\delta^2 = \frac{1}{K}\sum_{k=1}^{K} \frac{[\bar{D}_k - \bar{D}_{k-1} - \hat{\delta}(1-\bar{D}_{k-1})\Delta t]^2}{(1-\bar{D}_{k-1})^2(\Delta t)^3 k}, \tag{A.8}$$

where $\hat{\delta}$ is evaluated using (A.5). This idea is what Carroll and Ruppert (1988) call the *pseudo-likelihood method*.

Specification (A.8) gives an estimator for the variance of the sample average drawdown rate. Assuming, for simplicity, that the drawdown rates of the sample deals are perfectly positively correlated, it holds that $\hat{\sigma}_\delta^2 = \hat{\bar{\sigma}}_\delta^2$.

C. Distributions Rate

Following a similar approach to above, an appropriate goal function to estimate the distribution rate ν is given by

$$\sum_{k=1}^{K}(\bar{R}_k - E[\bar{R}_k|\mathcal{F}_{k-1}])^2, \tag{A.9}$$

where $\bar{R}_k$ denotes the sample average of the cumulated capital distributions at time t_k, i.e., $\bar{R}_k = \frac{1}{N}\sum_{j=1}^{N} R_{k,j}$, and $\mathcal{F}_{k-1}$ is the σ-field generated by $\bar{R}_1, \ldots, \bar{R}_{k-1}$.

The conditional expectation in (A.9) can be represented in discrete-time by:

$$E[\bar{R}_k|\mathcal{F}_{k-1}] = \bar{R}_{k-1} + \nu k \bar{V}_{k-1}(\Delta t)^2. \tag{A.10}$$

Substituting equation (A.10) into (A.9), the goal function to be minimized is given by

$$\sum_{k=1}^{K}\{\bar{R}_k - \bar{R}_{k-1} - \nu k\bar{V}_{k-1}(\Delta t)^2\}^2. \tag{A.11}$$

Similar to above, the least-squares estimator for ν is the solution to the equation $\sum_{k=1}^{K}(\partial/\partial\nu)\{\bar{R}_k - \bar{R}_{k-1} - \nu k\bar{V}_{k-1}(\Delta t)^2\}^2 = 0$. It turns out:

$$\hat{\nu} = \frac{1}{(\Delta t)^2}\frac{\sum\limits_{k=1}^{K}(\bar{R}_k - \bar{R}_{k-1})k\bar{V}_{k-1}}{\sum\limits_{k=1}^{K} k^2\bar{V}_{k-1}^2}. \tag{A.12}$$

The volatility σ_ν can be estimated by first noting that the conditional variance of the average capital distributions in the interval $(t_{k-1}, t_k]$ is given by

$$\begin{aligned} E[\bar{R}_k - E[\bar{R}_k|\mathcal{F}_{k-1}]|\mathcal{F}_{k-1}]^2 &= Var[\bar{\nu}_k\bar{V}_{k-1}\Delta t|\mathcal{F}_{k-1}] \\ &= (\bar{V}_{k-1}\Delta t)^2 Var[\bar{\nu}_k|\mathcal{F}_{k-1}]. \end{aligned} \tag{A.13}$$

Using specification (3.5), the conditional variance $Var[\bar{\nu}_k|\mathcal{F}_{k-1}]$ is given by:

$$Var[\bar{\nu}_k|\mathcal{F}_{k-1}] = \bar{\sigma}_\nu^2 k\Delta t. \tag{A.14}$$

Substituting equation (A.10) and (A.14) into (A.13), an appropriate estimator of the variance $\bar{\sigma}_\nu^2$ is

$$\hat{\bar{\sigma}}_\nu^2 = \frac{1}{K}\sum_{k=1}^{K}\frac{[\bar{R}_k - \bar{R}_{k-1} - \hat{\nu}k\bar{V}_{k-1}(\Delta t)^2]^2}{(\bar{V}_{k-1})^2(\Delta t)^3 k}, \tag{A.15}$$

where $\hat{\nu}$ is evaluated using (A.12).

Similar to above, specification (A.15) only gives an estimator for the variance of the sample average distribution rate. Assuming that the distribution rates of the sample deals are perfectly positively correlated, it also holds here that $\hat{\sigma}_\nu^2 = \hat{\bar{\sigma}}_\nu^2$.

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Tables and Figures

Table 1: Summary of Key Variables and Baseline Parameters

This table summarizes the symbols for the key variables in the model and baseline parameter values used for the numerical analysis. All model parameters are stated annualized.

Parameter	Notation	Value
Riskless rate	r_f	0.05
Expected return of stock market	μ_M	0.11
Total volatility of stock market returns	σ_M	0.15
Total volatility of PE fund returns	σ_V	0.40
Market beta of PE funds	β_V	1.30
Alpha of PE funds	α	0.04
Idiosyncratic volatility of PE fund returns	σ_ϵ	0.35
Return correlation between stock market and PE fund returns	ρ_V	0.49
Drawdown rate of PE funds	δ	0.41
Volatility of the drawdown rate	σ_δ	0.21
Correlation of the drawdown rate with stock market returns	ρ_δ	0.50
Average distribution rate	ν	0.08
Volatility of the distribution rate	σ_ν	0.11
Correlation of the distribution rate with stock market returns	ρ_ν	0.80
Long-run mean of secondary market discounts	θ_π	0.16
Mean-reversion speed of secondary market discounts	κ_π	0.42
Volatility of secondary market discounts	σ_π	0.16
Initial secondary market discount	π_0	0.28
Correlation between stock market returns and changes in secondary market discounts	ρ_c	-0.60
Life of PE fund investment (in years)	T_l	12

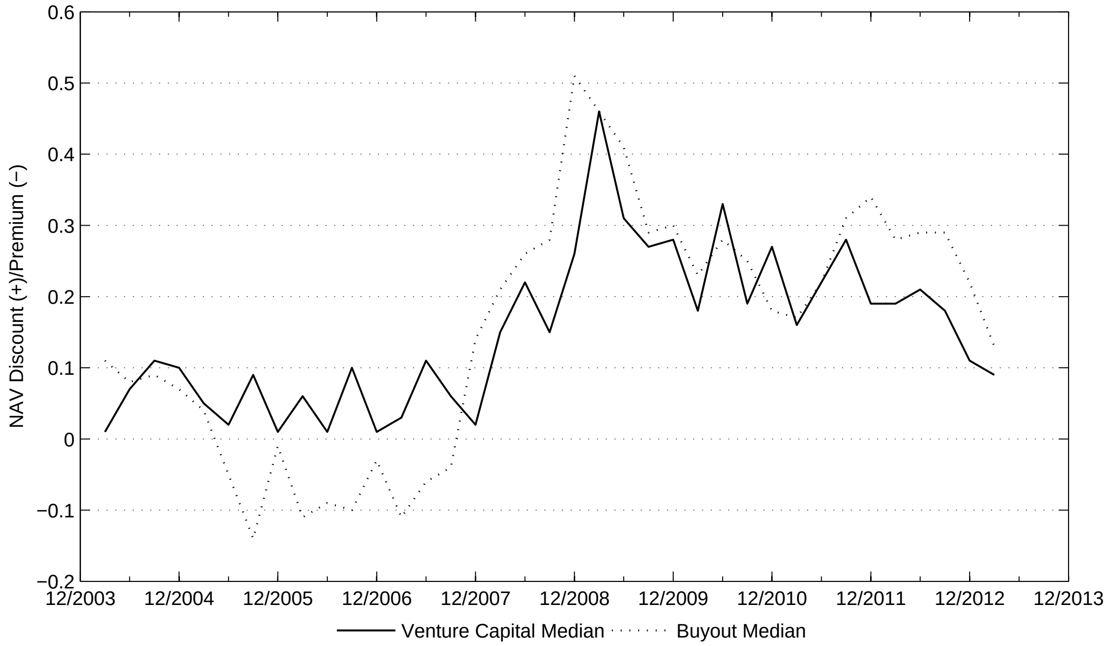
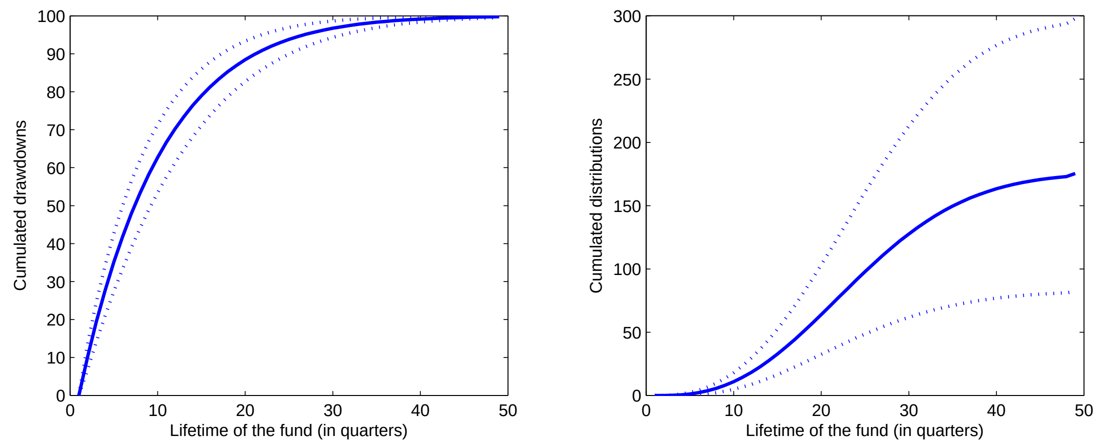
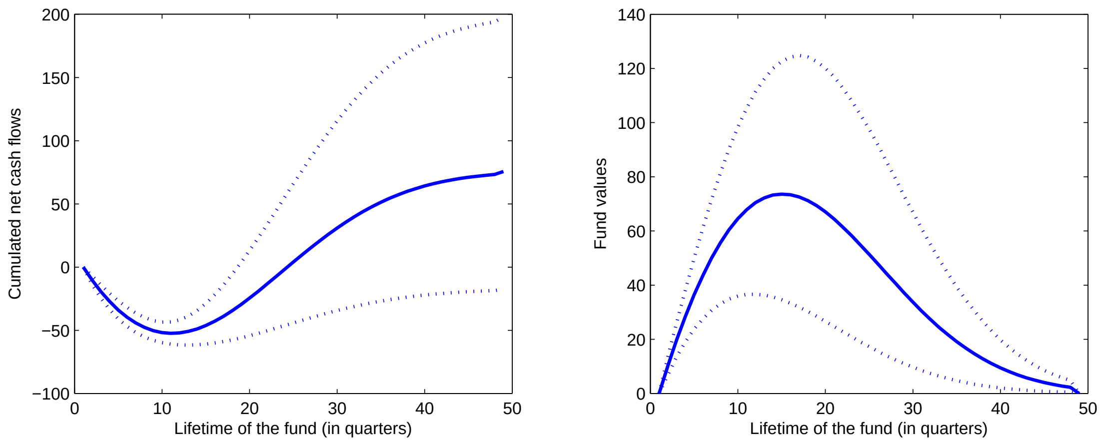


Figure 1: Median Discount (+)/Premium (-) to NAVs by Fund Type, 2004 - 2013; Data is obtained from the Preqin Secondary Market Monitor

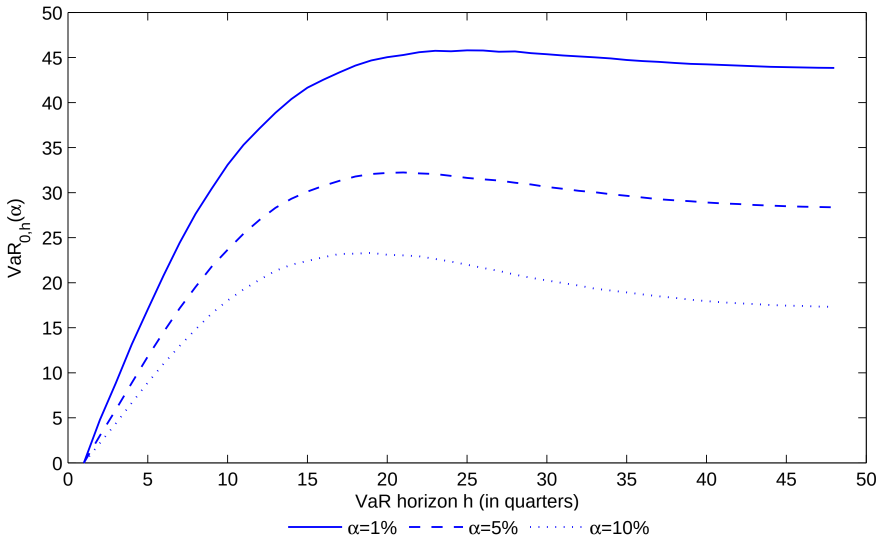


(a) Cumulated Capital Drawdowns (Left) and Cumulated Capital Distributions (Right)

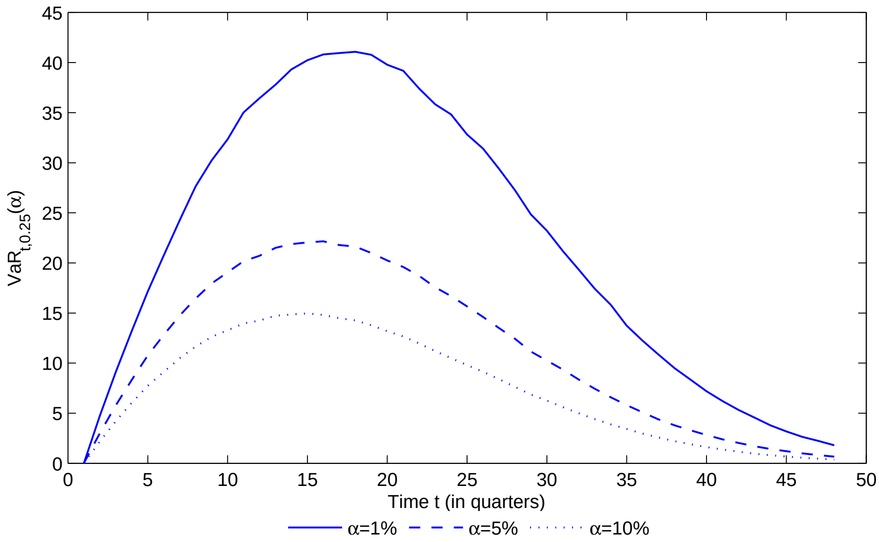


(b) Cumulated Net Fund Cash Flows (Left) and Fund Values (Right)

Figure 2: Cumulated Drawdowns, Repayments, Net Fund Cash Flows, and Fund Value Dynamics; Solid lines represent average values and dotted lines indicate 80% confidence intervals; Model parameters used are as given in Table 1

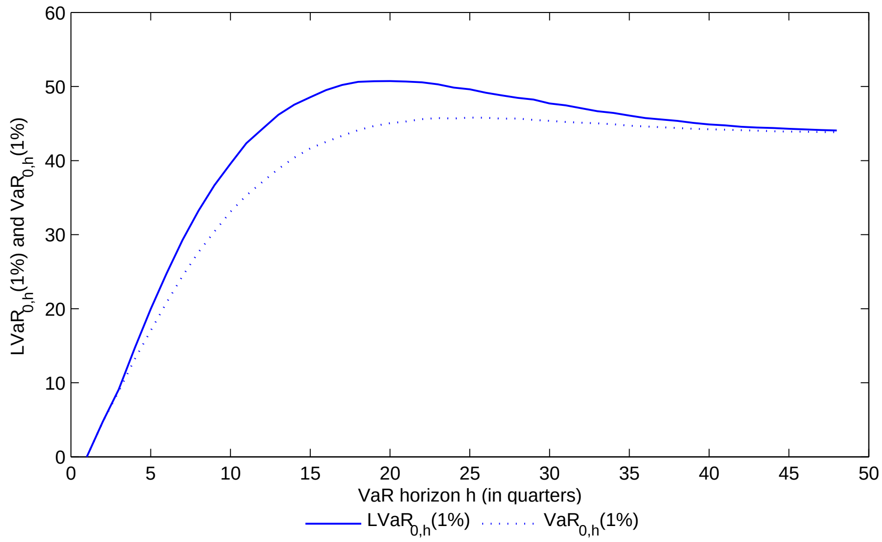


(a) VaR at fund initiation, $VaR_{0,h}(\alpha)$, plotted as a function of the time horizon h

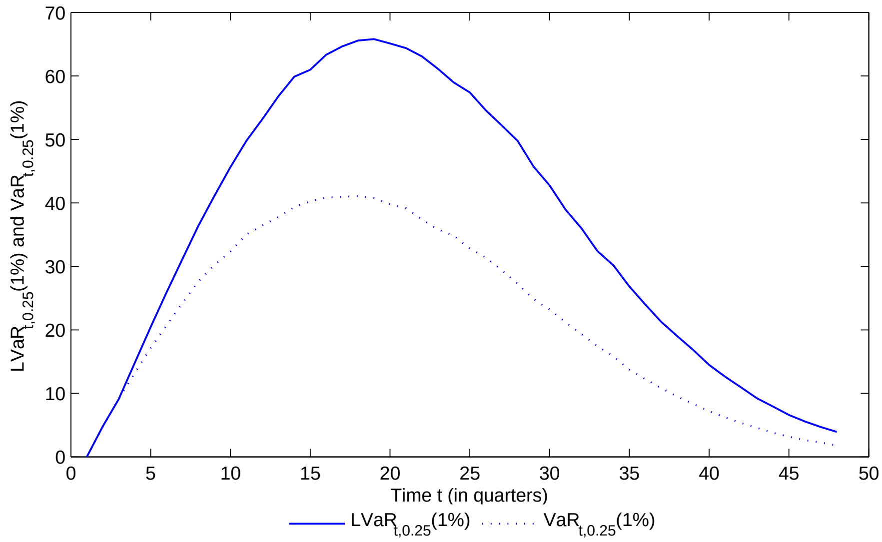


(b) Quarterly VaR, $VaR_{t,0.25}(\alpha)$, plotted as a function of time t

Figure 3: Value-at-Risk Dynamics over the Fund Lifecycle; Model parameters used are as given in Table 1

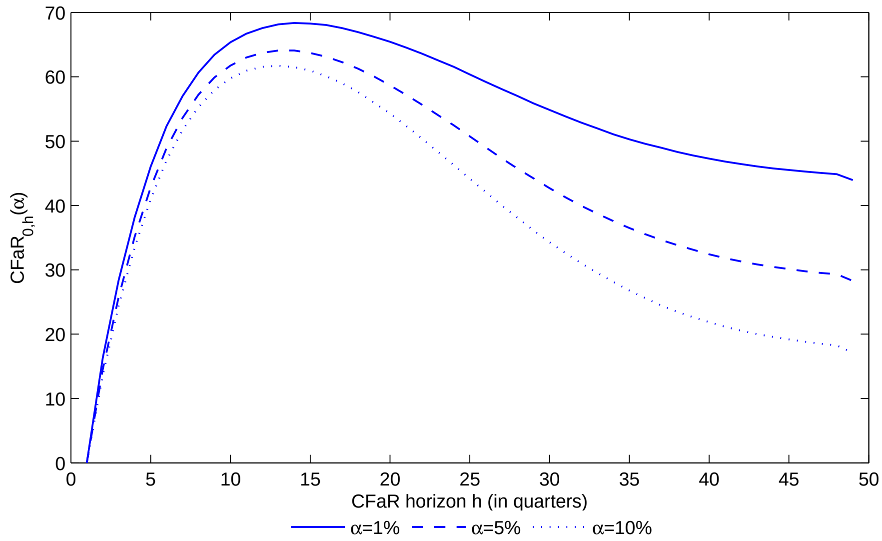


(a) $LVaR_{t,0.25}(\alpha)$ and $VaR_{t,0.25}(\alpha)$ plotted as functions of the time horizon h for $\alpha = 1\%$

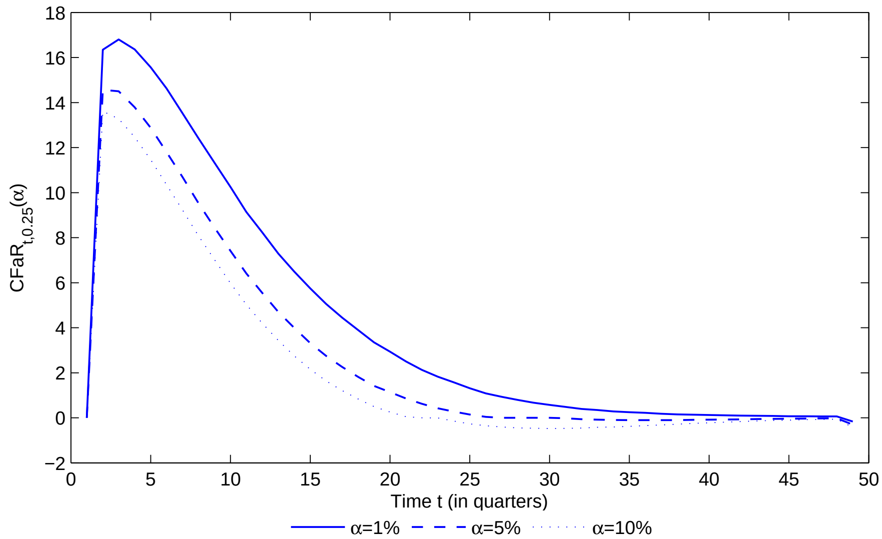


(b) $LVaR_{t,0.25}(\alpha)$ and $VaR_{t,0.25}(\alpha)$ plotted as functions of time t for $\alpha = 1\%$

Figure 4: Liquidity-Adjusted Value-at-Risk Compared to the Value-at-Risk for $\alpha = 1\%$; Model parameters used are as given in Table 1



(a) *CFaR* at fund initiation, $CFaR_{0,h}(\alpha)$, plotted as a function of the time horizon h



(b) Quarterly *CFaR*, $CFaR_{t,0.25}(\alpha)$, plotted as a function of time t

Figure 5: Cash-Flow-at-Risk Dynamics over the Fund Lifecycle; Model parameters used are as given in Table 1

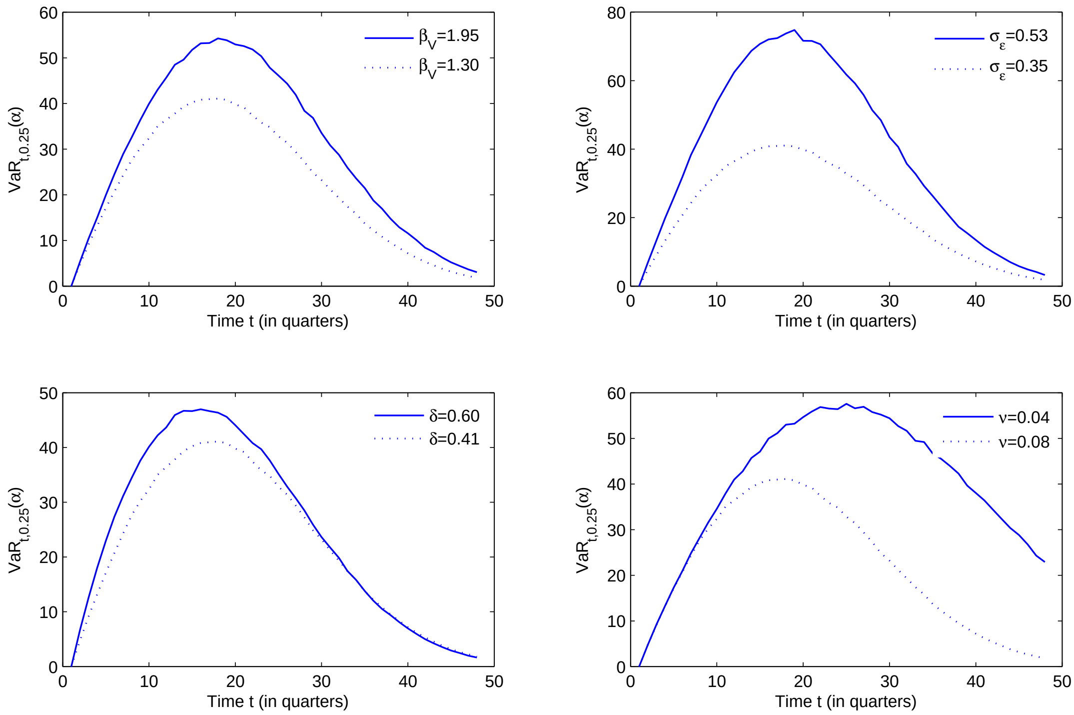


Figure 6: Sensitivity Analysis Value-at-Risk; Dotted lines show the baseline case and solid lines highlight the effect of parameter changes; The confidence level used is $\alpha = 1\%$; Other model parameters used are as given in Table 1

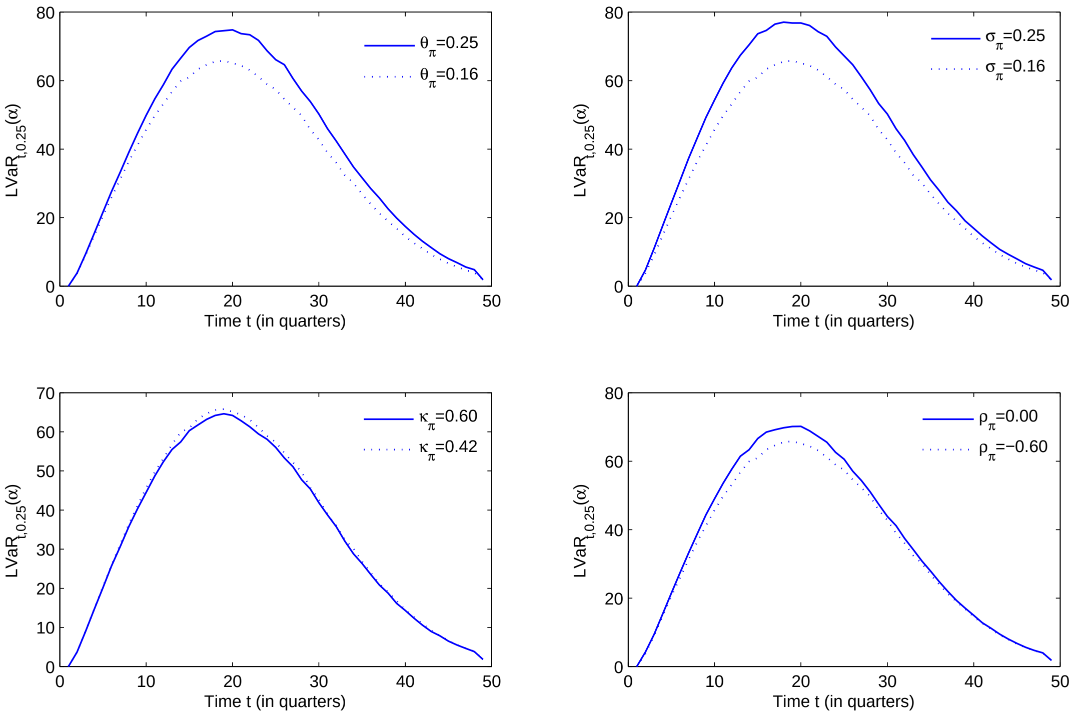


Figure 7: Sensitivity Analysis Liquidity-Adjusted Value-at-Risk; Dotted lines show the baseline case and solid lines highlight the effect of parameter changes; The confidence level used is $\alpha = 1\%$; Other model parameters used are as given in Table 1

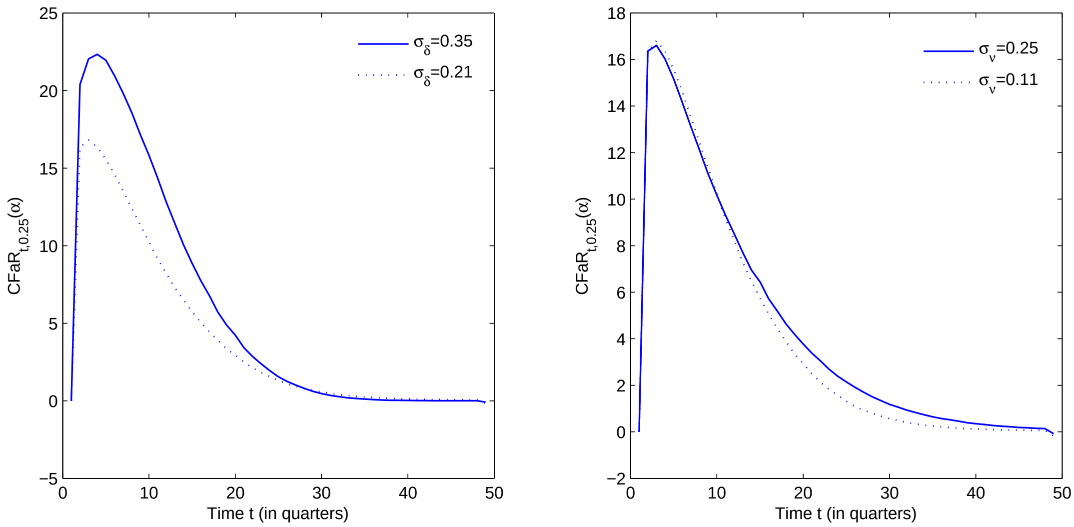


Figure 8: Sensitivity Analysis Cash-Flow-at-Risk; Dotted lines show the baseline case and solid lines highlight the effect of parameter changes; The confidence level used is $\alpha = 1\%$; Other model parameters used are as given in Table 1