

SEQUENCES

ARITHMETIC SEQUENCES

An ordered list of numbers such as: 4, 9, 16, 25, 36... is a **sequence**. Each number in the sequence is a **term**. Usually variables with subscripts are used to label terms. For example, in the sequence above, the first term is 4 and the third term is 16. This might be written $a_1 = 4$ and $a_3 = 16$ where a is the variable used to label the sequence.

In the sequence 1, 5, 9, 13, ..., there is a common difference ($d = 4$) between the successive terms and this is called an **arithmetic sequence**. There are two common methods to define a sequence. An explicit formula tells you exactly how to find any specific term in the sequence. A recursive formula tells first term and how to get from one term to the next. Formally, for arithmetic sequences, this is written:

Explicit: $a_n = a_1 + (n - 1)d$ where n = term number and d = common difference.

Recursive: a_1 = some specific value, $a_{n+1} = a_n + d$, and d = common difference.

For the sequence 1, 5, 9, 13, ..., the explicit formula is: $a_n = 1 + (n - 1)(4) = 4n - 3$ and the recursive formula is: $a_1 = 1$, $a_{n+1} = a_n + 4$. In each case, successively replacing n by 1, 2, 3, ... will yield the terms of the sequence. See the examples below.

Examples

List the first five terms of each arithmetic sequence.

Example 1 (An explicit formula)

$$a_n = 5n + 2$$

$$a_1 = 5(1) + 2 = 7$$

$$a_2 = 5(2) + 2 = 12$$

$$a_3 = 5(3) + 2 = 17$$

$$a_4 = 5(4) + 2 = 22$$

$$a_5 = 5(5) + 2 = 27$$

The sequence is: 7, 12, 17, 22, 27, ...

Example 2 (A recursive formula)

$$b_1 = 3, b_{n+1} = b_n - 5$$

$$b_1 = 3$$

$$b_2 = b_1 - 5 = 3 - 5 = -2$$

$$b_3 = b_2 - 5 = -2 - 5 = -7$$

$$b_4 = b_3 - 5 = -7 - 5 = -12$$

$$b_5 = b_4 - 5 = -12 - 5 = -17$$

The sequence is: 3, -2, -7, -12, -17, ...

Example 3 Find an explicit and a recursive formula for the sequence: -2, 1, 4, 7, ...

Explicit: $a_1 = -2$, $d = 3$ so the answer is: $a_n = a_1 + (n - 1)d = -2 + (n - 1)(3) = 3n - 5$

Recursive: $a_1 = -2$, $d = 3$ so the answer is: $a_1 = -2$, $a_{n+1} = a_n + 3$

Problems

List the first five terms of each arithmetic sequence.

1. $a_n = 5n - 2$

2. $b_n = -3n + 5$

3. $a_n = -15 + \frac{1}{2}n$

4. $c_n = 5 + 3(n - 1)$

5. $a_1 = 5, a_{n+1} = a_n + 3$

6. $a_1 = 5, a_{n+1} = a_n - 3$

7. $a_1 = -3, a_{n+1} = a_n + 6$

8. $a_1 = \frac{1}{3}, a_{n+1} = a_n + \frac{1}{2}$

Find the 30th term of each arithmetic sequence.

9. $a_n = 5n - 2$

10. $a_n = -15 + \frac{1}{2}n$

11. $a_{31} = 53, d = 5$

12. $a_1 = 25, a_{n+1} = a_n - 3$

For each arithmetic sequence, find an explicit and a recursive formula.

13. 4, 8, 12, 16, 20, ...

14. -2, 5, 12, 19, 26, ...

15. 27, 15, 3, -9, -21, ...

16. $3, 3\frac{1}{3}, 3\frac{2}{3}, 4, 4\frac{1}{3}, \dots$

Sequences are graphed using points of the form: (term number, term value).

For example, the sequence 4, 9, 16, 25, 36, ... would be graphed by plotting the points (1, 4), (2, 9), (3, 16), (4, 25), (5, 36), ... Sequences are graphed as points and not connected.

17. Graph the sequences from problems (1) and (2) above and determine the slope of each line.

18. How does the slope of the line found in the previous problem relate to the sequence?

GEOMETRIC SEQUENCES

In the sequence 2, 6, 18, 54, ..., there is a common ratio ($r = 3$) between the successive terms and this is called a **geometric sequence**. As before, there are two common methods to define a geometric sequence. The explicit formula tells you exactly how to find any specific term in the sequence. The recursive formula gives first term and how to get from one term to the next. Formally, for geometric sequences, this is written:

Explicit: $a_n = a_1 \cdot r^{n-1}$ where $n =$ term number and $r =$ common ratio

Recursive: $a_1 =$ some specific value and $a_{n+1} = a_n \cdot r$ where $r =$ common ratio

For the sequence 2, 6, 18, 54, ..., the explicit formula is: $a_n = a_1 \cdot r^{n-1} = 2 \cdot 3^{n-1}$, and the recursive formula is: $a_1 = 2$, $a_{n+1} = a_n \cdot 3$. In each case, successively replacing n by 1, 2, 3, ... will yield the terms of the sequence. See the examples below.

Examples

List the first five terms of each geometric sequence.

Example 1 (An explicit formula)

$$a_n = 3 \cdot 2^{n-1}$$

$$a_1 = 3 \cdot 2^{1-1} = 3 \cdot 2^0 = 3$$

$$a_2 = 3 \cdot 2^{2-1} = 3 \cdot 2^1 = 6$$

$$a_3 = 3 \cdot 2^{3-1} = 3 \cdot 2^2 = 12$$

$$a_4 = 3 \cdot 2^{4-1} = 3 \cdot 2^3 = 24$$

$$a_5 = 3 \cdot 2^{5-1} = 3 \cdot 2^4 = 48$$

The sequence is: 3, 6, 12, 24, 48, ...

Example 2 (A recursive formula)

$$b_1 = 8, \quad b_{n+1} = b_n \cdot \frac{1}{2}$$

$$b_1 = 8$$

$$b_2 = b_1 \cdot \frac{1}{2} = 8 \cdot \frac{1}{2} = 4$$

$$b_3 = b_2 \cdot \frac{1}{2} = 4 \cdot \frac{1}{2} = 2$$

$$b_4 = b_3 \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} = 1$$

$$b_5 = b_4 \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

The sequence is: 8, 4, 2, 1, $\frac{1}{2}$, ...

Example 3 Find an explicit and a recursive formula for the sequence: 81, 27, 9, 3, ...

Explicit: $a_1 = 81$, $r = \frac{1}{3}$ so the answer is: $a_n = a_1 \cdot r^{n-1} = 81 \cdot \left(\frac{1}{3}\right)^{n-1}$

Recursive: $a_1 = 81$, $r = \frac{1}{3}$ so the answer is: $a_1 = 81$, $a_{n+1} = a_n \cdot \frac{1}{3}$

Problems

List the first five terms of each arithmetic sequence.

19. $a_n = 5 \cdot 2^{n-1}$

20. $b_n = -3 \cdot 3^{n-1}$

21. $a_n = 40 \left(\frac{1}{2}\right)^{n-1}$

22. $c_n = 6 \left(-\frac{1}{2}\right)^{n-1}$

23. $a_1 = 5, a_{n+1} = a_n \cdot 3$

24. $a_1 = 100, a_{n+1} = a_n \cdot \frac{1}{2}$

25. $a_1 = -3, a_{n+1} = a_n \cdot (-2)$

26. $a_1 = \frac{1}{3}, a_{n+1} = a_n \cdot \frac{1}{2}$

Find the 15th term of each geometric sequence.

27. $b_{14} = 232, r = 2$

28. $b_{16} = 32, r = 2$

29. $a_{14} = 9, r = \frac{2}{3}$

30. $a_{16} = 9, r = \frac{2}{3}$

Find an explicit and a recursive formula for each geometric sequence.

31. 2, 10, 50, 250, 1250, ...

32. 16, 4, 1, $\frac{1}{4}$, $\frac{1}{16}$, ...

33. 5, 15, 45, 135, 405, ...

34. 3, -6, 12, -24, 48, ...

35. Graph the sequences from problems (19) and (32). Remember the note before problem (17) about graphing sequences.

36. How are the graphs of geometric sequences different from arithmetic sequences?

Answers

1. 3, 8, 13, 18, 23
2. 2, -1, -4, -7, -10
3. $-14\frac{1}{2}, -14, -13\frac{1}{2}, -13, -12\frac{1}{2}$
4. 5, 8, 11, 14, 17
5. 5, 8, 11, 14, 17
6. 5, 2, -1, -4, -7
7. -3, 3, 9, 15, 21
8. $\frac{1}{3}, \frac{5}{6}, 1\frac{1}{3}, 1\frac{5}{6}, 2\frac{1}{3}$
9. 148
10. 0
11. 48
12. -62
13. $a_n = 4n$; $a_1 = 4, a_{n+1} = a_n + 4$
14. $a_n = 7n - 9$; $a_1 = -2, a_{n+1} = a_n + 7$
15. $a_n = -12n + 39$; $a_1 = 27, a_{n+1} = a_n - 12$
16. $a_n = \frac{1}{3}n + 2\frac{2}{3}$; $a_1 = 3, a_{n+1} = a_n + \frac{1}{3}$
17. graph (1): linear points (1, 3), (2, 8), (3, 13), (4, 18), (5, 23) slope = 5
graph (2): linear points (1, 2), (2, -1), (3, -4), (4, -7), (5, -10) slope = -3
18. The slope of the line containing the points is the same as the common difference of the sequence.
19. 5, 10, 20, 40, 80
20. -3, -9, -27, -81, -243
21. 40, 20, 10, 5, $\frac{5}{2}$
22. 6, -3, $\frac{3}{2}, -\frac{3}{4}, \frac{3}{8}$
23. 5, 15, 45, 135, 405
24. 100, 50, 25, $\frac{25}{2}, \frac{25}{4}$
25. -3, 6, -12, 24, -48
26. $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}$
27. 464
28. 16
29. 6
30. $\frac{27}{2}$
31. $a_n = 2 \cdot 5^{n-1}$; $a_1 = 2, a_{n+1} = a_n \cdot 5$
32. $a_n = 16 \cdot \left(\frac{1}{4}\right)^{n-1}$; $a_1 = 16, a_{n+1} = a_n \cdot \frac{1}{4}$
33. $a_n = 5 \cdot 3^{n-1}$; $a_1 = 5, a_{n+1} = a_n \cdot 3$
34. $a_n = 3 \cdot (-2)^{n-1}$; $a_1 = 3, a_{n+1} = a_n \cdot (-2)$
35. Graph (19): Points on curve through (1, 5), (2, 10), (3, 20), (4, 40), and (5, 80).
Graph (32): Points on curve through (1, 16), (2, 4), (3, 1), (4, $\frac{1}{4}$), and (5, $\frac{1}{16}$).
36. Arithmetic sequences are linear and geometric sequences are curved (exponential).