

MATHCOUNTS Bible Completed

I. Squares and square roots: From 1^2 to 30^2 .

Squares: 1 - 50

1^2	=	1	26^2	=	676
2^2	=	4	27^2	=	729
3^2	=	9	28^2	=	784
4^2	=	16	29^2	=	841
5^2	=	25	30^2	=	900
6^2	=	36	31^2	=	961
7^2	=	49	32^2	=	1024
8^2	=	64	33^2	=	1089
9^2	=	81	34^2	=	1156
10^2	=	100	35^2	=	1225
11^2	=	121	36^2	=	1296
12^2	=	144	37^2	=	1369
13^2	=	169	38^2	=	1444
14^2	=	196	39^2	=	1521
15^2	=	225	40^2	=	1600
16^2	=	256	41^2	=	1681
17^2	=	289	42^2	=	1764
18^2	=	324	43^2	=	1849
19^2	=	361	44^2	=	1936
20^2	=	400	45^2	=	2025
21^2	=	441	46^2	=	2116
22^2	=	484	47^2	=	2209
23^2	=	529	48^2	=	2304
24^2	=	576	49^2	=	2401
25^2	=	625	50^2	=	2500

II. Cubes and cubic roots: From 1^3 to 12^3 .

Cubes: 1 -15

1^3	=	1	6^3	=	216	11^3	=	1331
2^3	=	8	7^3	=	343	12^3	=	1728
3^3	=	27	8^3	=	512	13^3	=	2197
4^3	=	64	9^3	=	729	14^3	=	2744
5^3	=	125	10^3	=	1000	15^3	=	3375

III. Powers of 2: From 2^1 to 2^{12} .

Powers of 2 and 3

2^1	=	2	3^1	=	3
2^2	=	4	3^2	=	9
2^3	=	8	3^3	=	27
2^4	=	16	3^4	=	81
2^5	=	32	3^5	=	243
2^6	=	64	3^6	=	729
2^7	=	128	3^7	=	2187
2^8	=	256	3^8	=	6561
2^9	=	512	3^9	=	19683
2^{10}	=	1024	3^{10}	=	59049
2^{11}	=	2048	3^{11}	=	177147
2^{12}	=	4096	3^{12}	=	531441

IV. **Prime numbers from 2 to 109:** It also helps to know the primes in the 100's, like 113, 127, 131, ... It's important to know not just the primes, but *why* 51, 87, 91, and others are *not primes*.

Prime numbers: 1 - 500

2	83	197	331	461
3	89	199	337	463
5	97	211	347	467
7	101	223	349	479
11	103	227	353	487
13	107	229	359	491
17	109	233	367	499
19	113	239	373	
23	127	241	379	
29	131	251	383	
31	137	257	389	
37	139	263	397	
41	149	269	401	
43	151	271	409	
47	157	277	419	
53	163	281	421	
59	167	283	431	
61	173	293	433	
67	179	307	439	
71	181	311	443	
73	191	313	449	
79	193	317	457	

- V. **Sum of the numbers in an arithmetic series:** In an *arithmetic series* the difference between terms is a constant. Example: $4 + 10 + 16 + 22 + \dots + 100$ is an arithmetic series. The formula for the sum is

$$n(a + z) / 2$$

where n is the number of terms in the sequence, a is the lowest term, and z is the highest term. Finding the sum of the above sequence:

$$17(4 + 100) / 2 = 17 * 104 / 2 = 884$$

Why is n equal to 17? Figure it out.

- VI. **Triangle or triangular numbers:** 3, 6, 10, 15, 21, 28, 36, 45, 55 are **triangle numbers**. (This is based on V. above!)

$$n(n + 1) / 2$$

To find $1 + 2 + 3 + 4 + 5 + \dots + n$, just take n , multiply it by its higher consecutive number, and divide by 2.

Example: find $1+2+3+4+5+\dots+28$.

$$28(28 + 1) / 2 = 406$$

These are the first 100 triangular numbers:

1	3	6	10	15	21	28	36	45	55
66	78	91	105	120	136	153	171	190	210
231	253	276	300	325	351	378	406	435	465
496	528	561	595	630	666	703	741	780	820
861	903	946	990	1035	1081	1128	1176	1225	1275
1326	1378	1431	1485	1540	1596	1653	1711	1770	1830
1891	1953	2016	2080	2145	2211	2278	2346	2415	2485
2556	2628	2701	2775	2850	2926	3003	3081	3160	3240
3321	3403	3486	3570	3655	3741	3828	3916	4005	4095
4186	4278	4371	4465	4560	4656	4753	4851	4950	5050

- VII. **Pythagorean Theorem:** Applications of this famous relationship occur very often in math competition and on the S.A.T.

$$a^2 + b^2 = c^2$$

- A. **Pythagorean Triples:** Integral values of a, b, and c, where a, b, and c are relatively prime:

3 - 4 - 5 (the most
common)

5 - 12 - 13

8 - 15 - 17

7 - 24 - 25

20 - 21 - 29

9 - 40 - 41

12 - 35 - 37

- B. **The 45° - 45° - 90° right triangle, or *right isosceles triangle*:** This is half of a square, where the legs are congruent. If the leg is s , then the *hypotenuse* is $s * \sqrt{2}$ (s times the square root of 2). Example: A square has a perimeter of 10, and you need to know the length of the diagonal.

$$s = 2.5, \text{ so } d = 2.5 * \sqrt{2}.$$

- C. **The 30° - 60° - 90° right triangle:** This is half of an *equilateral* triangle. The short leg, the one opposite the 30° angle, is s , the hypotenuse is $2s$, and the long leg, which is opposite the 60° angle, is $s * \sqrt{3}$ (s times the square root of 3).

- VIII. **Number of diagonals in an s -sided polygon:** I've seen so many different applications of this formula:

$$s(s - 3) / 2$$

where s is the number of sides of the polygon. A polygon having 45 sides has $45 * (45 - 3) / 2 = 945$ diagonals.

- IX. **Fraction, decimal, percent equivalencies:** You must know these backward, forward, and upside down. The *halves, thirds, fourths, fifths, sixths, sevenths*, (yes, sevenths!), *eighths, ninths, tenths*, (so hard, he?), *elevenths, twentieths, twenty-fifths, fiftieths*. It also helps to know the *twelfths, fifteenths, and sixteenths*. You should, for example, be able to recognize, instantly and without hesitation, that $83 \frac{1}{3}\%$ is $5/6$, and that $9/11$ is $81 \frac{9}{11}\%$.

X.

FRACTIONAL DECIMAL EQUIVALENTS							
1/64	0.015625	17/64	0.265625	33/64	0.515625	49/64	0.765625
1/32	0.03125	9/32	0.28125	17/32	0.53125	25/32	0.78125
3/64	0.046875	19/64	0.296875	35/64	0.546875	51/64	0.796875
1/16	0.0625	5/16	0.3125	9/16	0.5625	13/16	0.8125
5/64	0.078125	21/64	0.328125	37/64	0.578125	53/64	0.828125
3/32	0.09375	11/32	0.34375	19/32	0.59375	27/32	0.84375
7/64	0.109375	23/64	0.359375	39/64	0.609375	55/64	0.859375
1/8	0.125	3/8	0.375	5/8	0.625	7/8	0.875
9/64	0.140625	25/64	0.390625	41/64	0.640625	57/64	0.890625
5/32	0.15625	13/32	0.40625	21/32	0.65625	29/32	0.90625
11/64	0.171875	27/64	0.421875	43/64	0.671875	59/64	0.921875
3/16	0.1875	7/16	0.4375	11/16	0.6875	15/16	0.9375
13/64	0.203125	29/64	0.453125	45/64	0.703125	61/64	0.953125
7/32	0.21875	15/32	0.46875	23/32	0.71875	31/32	0.96875
15/64	0.234375	31/64	0.484375	47/64	0.734375	63/64	0.984375
1/4	0.250	1/2	0.500	3/4	0.750	1	1.000

XI. **Space diagonal of a cube:** $s * \text{sqrt}(3)$

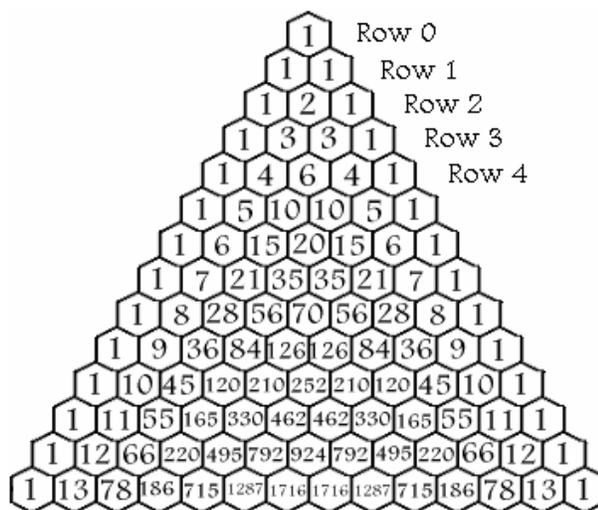
where s is the edge of the cube. This is an application of the Pythagorean Theorem: See Section VII(B) above. Figure out why this is so. Don't expect me to do it for you.

XII. **Area and Volume:**

- A. Area of a square, given the side: $A = s^2$
- B. Area of a square, given diagonal: $A = d^2/2$
- C. Area of a rhombus, given diagonals: $A = (d_1 d_2)/2$
(B and C are closely related. How?)
- D. Area of triangle: $A = (bh) / 2$
- E. Area of circle: $A = r^2$
- F. Area of trapezoid: $A = 1/2 h (b_1 + b_2)$
- G. Volume of cylinder and prism: $V = B h$
- H. Volume of cone and pyramid: $V = 1/3 (B h)$
- I. Volume of a sphere: $V = 4/3 r^3$
- J. Surface area of a sphere: $A = 4 r^2$

XIII. **I also expect you to know the following procedures:**

- A. *Scientific notation*, both multiplying and dividing numbers written in this form. All you do is apply the rules you've learned about *exponents*.
- B. Turning a *repeating decimal* into a *simple fraction*. You see this almost every week; isn't it time to learn the shortcut for this, once and for all?
- C. Turning a *fractional percent* into a simple fraction. Example: $20 \frac{5}{6}\% = \frac{5}{24}$
- D. Setting up *probability problems*. This is usually plain, simple reading. Know the terms "with replacement"; "without replacement", "at least one".
- E. Be able to generate *Pascal's Triangle* on the spot. There are so many applications of this in *combinations* and *probability*.



If a row is made into a single number by using each element as a digit of the number (carrying over when an element itself has more than one digit), the number is equal to 11 to the n^{th} power or 11^n when n is the number of the row the multi-digit number was taken from.

Row #	Formula	=	Multi-Digit number	Actual Row
Row 0	11^0	=	1	1
Row 1	11^1	=	11	1 1
Row 2	11^2	=	121	1 2 1
Row 3	11^3	=	1331	1 3 3 1
Row 4	11^4	=	14641	1 4 6 4 1
Row 5	11^5	=	161051	1 5 10 10 5 1
Row 6	11^6	=	1771561	1 6 15 20 15 6 1
Row 7	11^7	=	19487171	1 7 21 35 35 21 7 1
Row 8	11^8	=	214358881	1 8 28 56 70 56 28 8 1

- F. Can you think of any more? I can. You should.

More on Triangular Numbers

The smallest square numbers

$$\begin{aligned}
 1 &= 1^2 \\
 d_8 &= 36 = 6^2 \\
 d_{49} &= 1225 = 35^2 \\
 d_{288} &= 41616 = 204^2 \\
 d_{1681} &= 1413721 = 1198^2 \\
 d_{9800} &= 480024900 = 6930^2 \\
 d_{57121} &= 1631432881 = 40391^2 \\
 &\dots
 \end{aligned}$$

The smallest palindromic numbers

$$\begin{aligned}
 d_{10} &= 55 \\
 d_{11} &= 66 \\
 d_{18} &= 171 \\
 d_{34} &= 595 \\
 d_{36} &= 666 \\
 d_{77} &= 3003 \\
 d_{109}, d_{132}, d_{173}, d_{363}, & \\
 &\dots
 \end{aligned}$$

Perfect numbers

A number which is equal to the sum of all its divisors smaller than the number itself is called a perfect number.

The first perfect numbers are 6, 28 and 496. They are triangular numbers like every perfect number.

666 is the largest triangular number which you can form of the same digits

- The sum of two consecutive triangular numbers is always a square:

$$\begin{aligned}
 T_1 + T_2 &= 1 + 3 = 4 = 2^2 \\
 T_2 + T_3 &= 3 + 6 = 9 = 3^2
 \end{aligned}$$

- If T is a Triangular number then $9 \cdot T + 1$ is also a Triangular number:

$$\begin{aligned}
 9 \cdot T_1 + 1 &= 9 \cdot 1 + 1 = 10 = T_4 \\
 9 \cdot T_2 + 1 &= 9 \cdot 3 + 1 = 28 = T_{17}
 \end{aligned}$$

- A Triangular number can never end in 2, 4, 7 or 9:
- If T is a Triangular number then $8 \cdot T + 1$ is always a perfect square:

$$\begin{aligned}
 8 \cdot T_1 + 1 &= 8 \cdot 1 + 1 = 9 = 3^2 \\
 8 \cdot T_2 + 1 &= 8 \cdot 3 + 1 = 25 = 5^2
 \end{aligned}$$

- The digital root (i.e. ultimate sum of digits until a single digit is obtained) of triangular numbers is always 1, 3, 6 or 9.
- The sum of n consecutive cubes starting from 1 is equal to the square of n^{th} triangular number i.e. $T_n^2 = 1^3 + 2^3 + 3^3 + \dots + n^3$

$$\begin{aligned}
 T_4^2 &= 10^2 = 100 = 1^3 + 2^3 + 3^3 + 4^3 \\
 T_5^2 &= 15^2 = 225 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3
 \end{aligned}$$

- A triangular number greater than 1, can never be a Cube, a Fourth Power or a Fifth power.
- All perfect numbers are triangular numbers
- The square of triangular numbers 1 and 6 produce triangular numbers 1 and 36.

$$T_1^2 = 1 * 1 = 1 = T_1$$

$$T_3^2 = 6 * 6 = 36 = T_8$$

- The only triangular number which is also a prime is 3.
- **Palindromic Triangular Numbers:** Some of the many triangular numbers, which are also palindromic (i.e. reading the same forward as well as backward) are 1, 3, 6, 55, 66, 171, 595, 666, 3003, 5995, 8778, 15051, 66066, 617716, 828828, 1269621, 1680861, 3544453, 5073705, 5676765, 6295926, 351335153, 61477416, 178727871, 1264114621, 1634004361 etc. These can be termed as **palindromic triangular numbers**. There are 28 **Palindromic Triangular numbers** below 10^{10} . For more on these numbers visit [Patrick De Geest](#)
- **Reversible Triangular Numbers:** Some of the many triangular numbers, whose reversals are also triangular numbers are 1,3,6,10,55,66,120,153,171,190,300,351,595,630,666,820,3003,5995,8778,15051, 17578,66066,87571,156520,180300,185745,547581,557040,617716,678030,828828, 1269621,1461195,1680861,1851850,3544453,5073705,5676765,5911641,6056940, 6295926,12145056,12517506,16678200,35133153,56440000,60571521, 61477416, 65054121,157433640,178727871,188267310,304119453,354911403,1261250200, 1264114621,1382301910,1634004361,1775275491,1945725771 etc. These can be termed as **Reversible triangular numbers**. Note that all **palindromic triangular numbers** mentioned above are special case of **reversible triangular numbers**.
- There are pairs of triangular numbers such that the sum and difference of numbers in each pair are also triangular numbers e.g. (15, 21), (105, 171), (378, 703), (780, 990), (1485, 4186), (2145, 3741), (5460, 6786), (7875, 8778)... etc.:

$$21 + 15 = 36 = T_8 : 21 - 15 = 6 = T_3$$

$$171 + 105 = 276 = T_{23} : 171 - 105 = 66 = T_{11}$$

$$703 + 378 = 1081 = T_{46} : 703 - 378 = 325 = T_{25}$$

- There are some triangular numbers which are product of three consecutive numbers. For example 120 is a triangular number which is product of three consecutive numbers 5, 6 and 7. There are only 6 such triangular numbers, largest of which is 258474216, as shown below:

$$1 * 2 * 3 = 6 = T_3$$

$$4 * 5 * 6 = 120 = T_{15}$$

$$5 * 6 * 7 = 210 = T_{20}$$

$$9 * 10 * 11 = 990 = T_{44}$$

$$56 * 57 * 58 = 185136 = T_{608}$$

$$636 * 637 * 638 = 258474216 = T_{22736}$$

- The triangular number 120 is the product of three, four and five consecutive numbers.

$$4 * 5 * 6 = 2 * 3 * 4 * 5 = 1 * 2 * 3 * 4 * 5 = 120$$

No other triangular number is known to be the product of four or more consecutive numbers.

- There are some triangular numbers which are product of two consecutive numbers. For example 6 is a triangular number which is product of two consecutive numbers 2 and 3. Some others are as shown below:

$$2 * 3 = 6 = T_3$$

$$14 * 15 = 210 = T_{20}$$

$$84 * 85 = 7140 = T_{119}$$

$$492 * 493 = 242556 = T_{696}$$

$$2870 * 2871 = 8239770 = T_{4059}$$

$$16730 * 16731 = 279909630 = T_{23660}$$

$$97512 * 97513 = 9508687656 = T_{137903}$$

$$568344 * 568345 = 323015470680 = T_{803760}$$

$$3312554 * 3312555 = 10973017315470 = T_{4684659}$$

$$19306982 * 19306983 = 372759573255306 = T_{27304196}$$

- The triangular numbers which are product of two prime numbers can be termed as **Triangular Semiprimes**. For example 6 is a **Triangular Semiprimes**. Some other examples of **Triangular Semiprimes** are: 10,15,21,55,91,253,703,1081,1711,1891,2701,3403,5671,12403,13861,15931, 18721,25651,34453,38503,49141,60031,64261,73153,79003,88831,104653,108811, 114481,126253,146611,158203,171991,188191,218791,226801,258121, 269011,286903, 351541,371953,385003,392941,482653,497503 etc as shown below:

$$2 * 3 = 6 = T_3$$

$$3 * 5 = 15 = T_5$$

$$3 * 7 = 21 = T_6$$

$$5 * 11 = 55 = T_{10}$$

$$7 * 13 = 91 = T_{13}$$

$$11 * 23 = 253 = T_{22}$$

$$19 * 37 = 703 = T_{37}$$

- Harshad Triangular Numbers:**

Harshad (or Niven) numbers are those numbers which are divisible by their sum of the digits. For example 1729 (19*91) is divisible by 1+7+2+9 =19, so 1729 is a Harshad number.

Harshad Triangular Number can be defined as the Triangular numbers which are divisible by the sum of their digits. For example, Triangular number 1128 is divisible by $1+1+2+8 = 12$ (i.e. $1128/12 = 94$). So 1128 is a **Harshad Triangular Number**. Other examples are:

1, 3, 6, 10, 21, 36, 45, 120, 153, 171, 190, 210, 300, 351, 378, 465, 630, 666, 780, 820, 990, 1035, 1128, 1275, 1431, 1540, 1596, 1770, 2016, 2080, 2556, 2628, 2850, 2926, 3160, 3240, 3321, 3486, 3570, 4005, 4465, 4560, 4950, 5050, 5460, 5565, 5778, 5886, 7140, 7260, 8001, 8911, 9180, 10011, 10296, 10440, 11175, 11476, 11628, 12720, 13041, 13203, 14196, 14706, 15225, 15400, 15576, 16110, 16290, 16653, 17020, 17205, 17766, 17955, 18145, 18528, 20100, 21321, 21528, 21736, 21945, 22155, 23220, 23436, 24090, 24310, 24976, 25200, 28680, 29646, 30628, 31626, 32640, 33930, 35245, 36585, 37128, 39060, 40470, 41328, 41616, 43365, 43956, 45150, 46360, 51040, 51360, 51681, 52326, 52650, 53956, 56280, 56616, 61776, 63903, 64620, 65341, 67896, 69006, 70125, 70500, 72010, 73536, 73920, 76636, 78210, 79401, 79800, 80200, 81810, 88410, 89676, 90100, 93096, 93528, 97020, 100128, 101025, 103740, 105111, 105570 etc.

- **Happy Triangular Numbers:**

If you iterate the process of summing the squares of the decimal digits of a number and if the process terminates in 1, then the original number is called a **Happy number**. For example $7 \rightarrow 49 \rightarrow 97 \rightarrow 130 \rightarrow 10 \rightarrow 1$.

A Happy Triangular Number is defined as a Triangular number which is also a Happy number. For example, consider a triangular number 946, where $946 \rightarrow 133 \rightarrow 19 \rightarrow 82 \rightarrow 68 \rightarrow 100 \rightarrow 1$. So 946 is a **Happy triangular Number**. Other examples of Happy Triangular Numbers are :

1, 10, 28, 91, 190, 496, 820, 946, 1128, 1275, 2080, 2211, 2485, 3321, 4278, 8128, 8256, 8778, 9591, 9730, 11476, 12090, 12880, 13203, 13366, 13530, 15753, 16471, 17205, 17578, 20910, 21115, 21321, 22791, 24753, 25651, 27261, 29890, 30135, 31626, 33670, 35245, 36046, 41328, 43660, 43956, 44253, 46360, 47586, 48205, 50721, 53301, 53956, 54615, 55278, 56280, 56953, 58311, 61425, 62128, 66430, 69378, 69751, 70125, 75855, 76245, 77815, 79003, 80200, 81810, 82621, 84666, 87571, 90100, 90951, 93961, 99681, 100128, 101025, 102831, 103285, 105570, 107416, 110215, 117370, 119316, 122760, 123256, 123753, 126253, 127260, 129286, 130305 etc.

- **The only Fibonacci Numbers that are also triangular are 1, 3, 21 and 55.**
- **The only triangular Numbers that are also repdigit are 55, 66 and 666.**