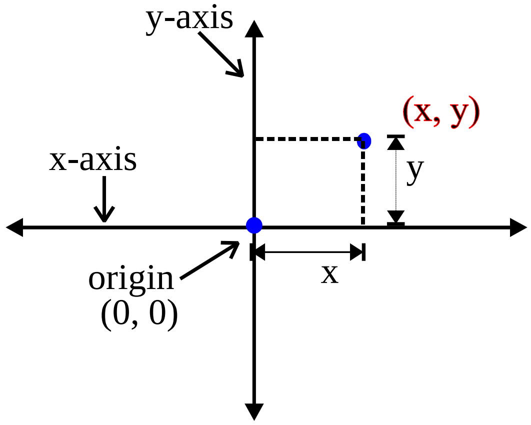
Chapter 3A - Rectangular Coordinate System

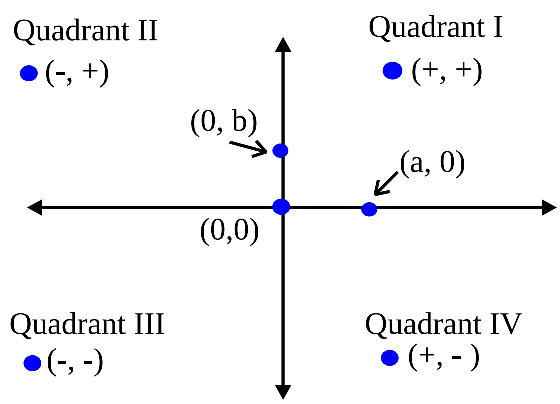
Introduction: Rectangular Coordinate System

Although the use of rectangular coordinates in such geometric applications as surveying and planning has been practiced since ancient times, it was not until the 17th century that geometry and algebra were joined to form the branch of mathematics called analytic geometry. French mathematician and philosopher Rene Descartes (1596-1650) devised a simple plan whereby two number lines were intersected at right angles with the position of a point in a plane determined by its distance from each of the lines. This system is called the rectangular coordinate system (or Cartesian coordinate system).



Points are labeled with ordered pairs of real numbers (x, y), called the coordinates of the point, which give the horizontal and vertical distance of the point from the origin, respectively. The origin is the intersection of the *x*- and *y*-axes. Locations of the points in the plane are determined in relationship to this point $(0,0)$. All points in the plane are located in one of four quadrants or on the *x*- or *y*-axis as illustrated below.

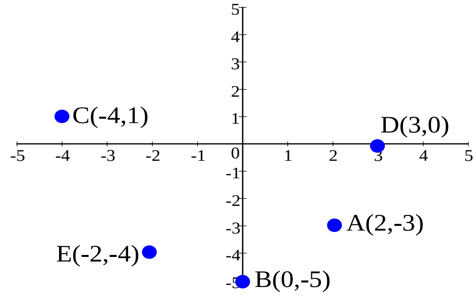
To plot a point, start at the origin, proceed horizontally the distance and direction indicated by the *x*-coordinate, then vertically the distance and direction indicated by the *y*-coordinate. The resulting point is often labeled with its ordered pair coordinates and/or a capital letter. For example, the point 2 units to the right of the origin and 3 units up could be labeled $A(2,3)$.



Notice that the Cartesian plane has been divided into fourths. Each of these fourths is called a quadrant and they are numbered as indicated above.

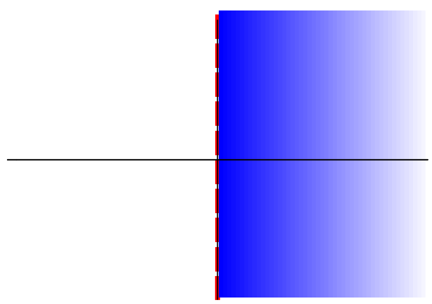
Example 1: Plot the following points on a rectangular coordinate system:
$A(2,-3)$ $B(0,-5)$ $C(-4,1)$ $D(3,0)$ $E(-2,-4)$

Solution:



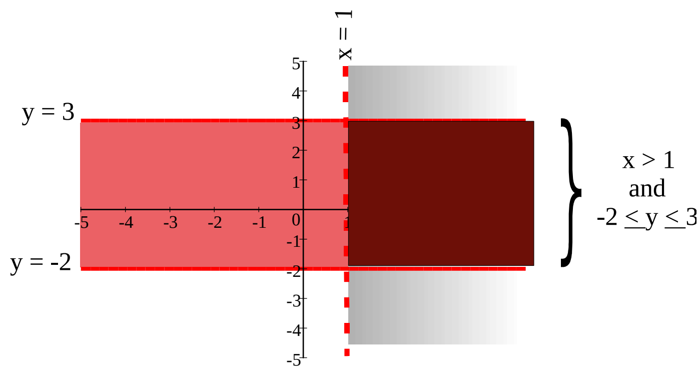
Example 2: Shade the region of the coordinate plane that contains the set of ordered pairs $\{(x,y) \mid x > 0\}$. [The set notation is read "the set of all ordered pairs (x,y) such that $x > 0$".]

Solution: This set describes all ordered pairs where the *x*-coordinate is greater than 0. Plot several points that satisfy the stated condition, e.g., $(2,-4)$, $(7,3)$, $(4,0)$. These points are all located to the right of the *y*-axis. To plot all such points we would shade all of Quadrants I and IV. We indicate that points on the *y*-axis are not included ($x = 0$) by using a dotted line.



Example 3: Shade the region of the coordinate plane that contains the set of ordered pairs $\{(x,y) \mid x > 1, -2 \le y \le 3\}$.

Solution: The area to the right of the dotted line designated $x = 1$ is the set of all points where the *x*-coordinate is greater than 1 (shaded gray). The area between the horizontal lines designated $y = -2$ and $y = 3$ is the area where the *y*-coordinate is between -2 and 3 (shaded red). The dark region is the intersection of these two sets of points, the set that satisfies both of the given conditions.

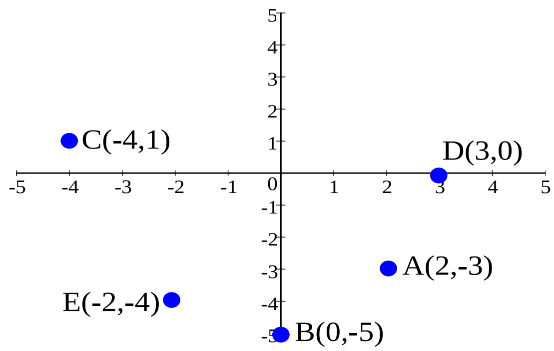


The basis of analytic geometry lies in the connection between a set of ordered pairs and its graph on the Cartesian coordinate system.

Definitions:

- Any set of ordered pairs is called a **relation**.
- The plot of every point associated with an ordered pair in the relation is called the **graph** of the relation.
- The set of all first elements in the ordered pairs is called the **domain** of the relation.
- The set of all second elements in the ordered pairs is called the **range** of the relation.

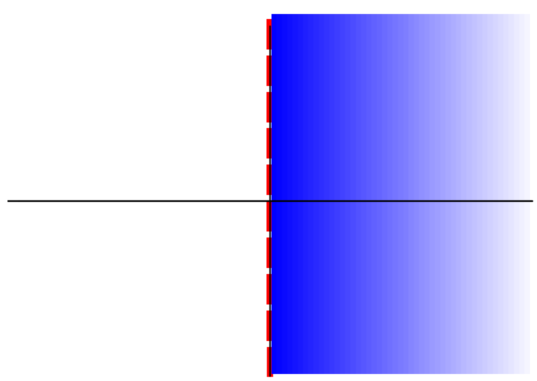
In Example 1, we plotted five distinct points. If we consider these points as a set of ordered pairs, we have the relation $\{(2,-3),\ (0,-5),\ (-4,1),\ (3,0),\ (-2,-4)\}$.
The graph is



The **domain** is $\{2,\ 0,\ -4,\ 3,\ -2\}$ and the **range** is $\{-3,\ -5,\ 1,\ 0,\ -4\}$.

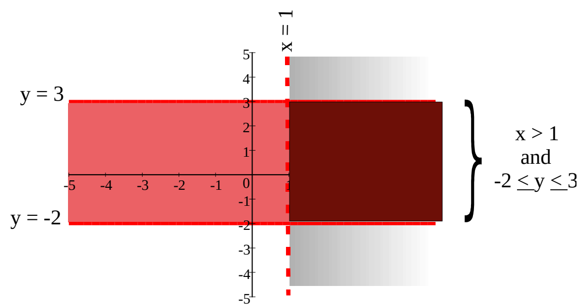
Infinite sets of ordered pairs can be described algebraically and plotted (or graphed) on the coordinate system.

Example 4: Below is the graph from Example 2. Recall that the graph represents all ordered pairs defined by the algebraic statement: $x > 0$. That is, the relation consists of all ordered pairs (x, y) that have an x-coordinate that is a positive number. What is the domain and range of this relation?



Solution: Since the relation is defined as the set of all ordered pairs where $x > 0$, the domain is $x > 0$. The y-coordinates can be any real number so the range is *all* real *numbers*.

Example 5: Below is the graph from Example 3. What is the domain and range?



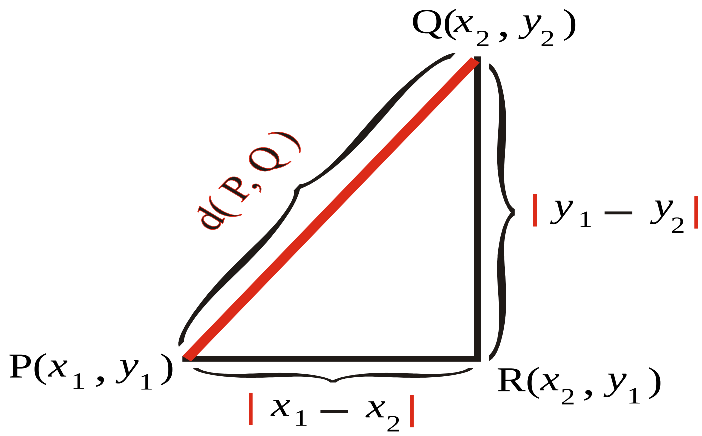
Solution: The domain is all real numbers greater than 1. The range is all real numbers between -2 and 3, including the endpoints -2 and 3.

Note: We often write the domain and range in interval notation. The domain for the above example in interval notation is $(1,\infty)$. The range for the above example in interval notation is $[-2,3]$.

Distance Formula

The marriage of algebra and geometry allows us to devise algebraic formulas to use in solving geometric problems. For example, the formula for finding the distance between two points in the plane is derived as follows:

Consider two points $P(x_1, y_1)$ and $Q(x_2, y_2)$. Select a third point $R(x_2, y_1)$ so that the three points form a right triangle with the right angle at point R. (See figure below.)



Note that the distance between P and R is $|x_2 - x_1|$ and the distance from Q and R is $|y_2 - y_1|$. Therefore, the distance from point P to point Q , denoted $d(P, Q)$, can be found using the Pythagorean Theorem:

$$(d(P, Q))^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

Because $|x_2 - x_1|^2 = (x_2 - x_1)^2$ and $|y_2 - y_1|^2 = (y_2 - y_1)^2$, and because we are only interested in positive values for the distance, we obtain the following formula.

Distance Formula

The **distance** between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the plane is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

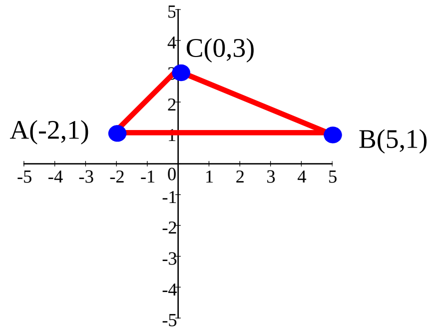
Example 1: Find the distance between the points $A(3, -2)$ and $B(-4, -7)$.

Solution: To find the distance between two points in the plane we use the distance formula $d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. It does not matter which point you use as (x_1, y_1) or (x_2, y_2), so we will use the coordinates of A and B respectively. That is, $x_1 = 3$, $y_1 = -2$, $x_2 = -4$, and $y_2 = -7$. Plugging into the formula, we get

$$d(A, B) = \sqrt{(-4-3)^2 + (-7-(-2))^2} = \sqrt{(-7)^2 + (-5)^2} = \sqrt{74}\,.$$

Example 2: Determine whether the points $A(-2,1)$, $B(5,1)$ and $C(0,3)$ are the vertices of a right triangle.

Solution: Plot and label points *A*, *B*, and *C* on graph paper and draw the indicated triangle.



From inspection of the graph, point *C* appears to be the vertex of the right triangle; therefore the line segment through *A* and *B* appears to be the hypotenuse. We will use the distance formula to find the length of each side of the triangle, and then apply the Pythagorean Theorem to determine whether the triangle is indeed a right triangle:

$$d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d(A,B) = \sqrt{(5-(-2))^2 + (1-1)^2} = \sqrt{(7)^2 + (0)^2} = \sqrt{49} = 7$$

$$d(B,C) = \sqrt{(5-0)^2 + (1-3)^2} = \sqrt{(5)^2 + (-2)^2} = \sqrt{25+4} = \sqrt{29}$$

$$d(A,C) = \sqrt{(0-(-2))^2 + (3-1)^2} = \sqrt{2^2 + 2^2} = \sqrt{8}$$

If triangle *ABC* is a right triangle,

$$\left(\sqrt{8}\right)^2 + \left(\sqrt{29}\right)^2 = (7)^2$$

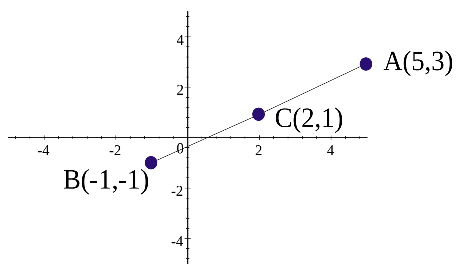
$$8 + 29 = 49$$

$$37 = 49$$

must be true. Since $37 \neq 49$, triangle *ABC* is NOT a right triangle.

Example 3: Determine whether the points $A(5,3)$and $B(-1,-1)$ are equidistant from point $C(2,1)$. Recall that equidistant means "equal distance". That is, points *A* and *B* are equidistant from point *C* if and only if the distance from *A* to *C* is equal to the distance from *B* to *C*. Algebraically, we would write the above statement as $d(A,C) = d(B,C)$.

Solution: Plot the points on graph paper to visualize the problem.



Using the distance formula we can find the distance between each of the points, A and B, and point C.

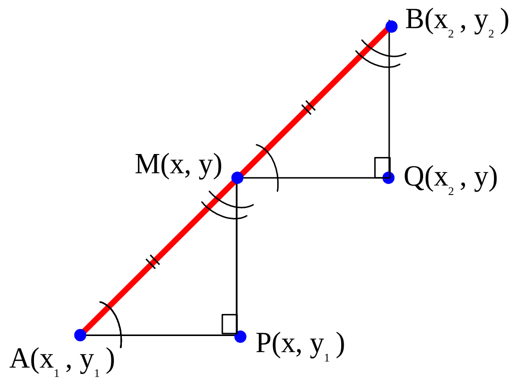
$$d(A,C) = \sqrt{(5-2)^2 + (3-1)^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$d(B,C) = \sqrt{(-1-2)^2 + (-1-1)^2} = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$$

Because $d(A,C) = d(B,C)$, *A* and *B* are equidistant from point *C*. Note that if you plot all the points that are $\sqrt{13}$ units from point *C* you will obtain a circle.

Midpoint Formula

In many instances it is important to be able to calculate the point that lies half way between the two points on the line segment that connects them. The figure below shows points $A(x_1, y_1)$ and $B(x_2, y_2)$, along with their midpoint $M(x, y)$.



Note that $\Delta APM \simeq \Delta BQM$ by ASA so that $\overline{AP} = \overline{MQ}$.

$$x - x_1 = x_2 - x$$

Solving for x, we get

$$2x = x_2 + x_1$$
$$x = \frac{x_2 + x_1}{2}.$$

Similarly,

$$y = \frac{y_2 + y_1}{2}.$$

Midpoint Formula

The midpoint of the line segment that connects points $A(x_1, y_1)$ and $B(x_2, y_2)$ is the point $M(x, y)$ with coordinates $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$. The midpoint is the point $M(x, y)$ with $x = \frac{x_1 + x_2}{2}$ and $y = \frac{y_1 + y_2}{2}$.

Note that the x-coordinate of the midpoint is the average of the x-coordinates of the the endpoints, and the y-coordinate of the midpoint is the average of the y-coordinates of the endpoints.

The examples below will show you how to apply this formula. Be sure to work through them before trying the exercises.

Example 1: Find the midpoint of the line segment that connects the points $(-2, 9)$ and $(7, -3)$.

Solution: Plugging $x_1 = -2$, $y_1 = 9$, $x_2 = 7$, and $y_2 = -3$ into the mid-point formula, we get

$$x = \frac{-2 + 7}{2} = \frac{5}{2} \qquad y = \frac{9 + (-3)}{2} = 3$$

Therefore, the midpoint is $\left(\frac{5}{2}, 3\right)$.

Example 2: If $(-5, 8)$ is the midpoint of the line segment connecting $A(3,-2)$ and B, find the coordinates of the other endpoint B.

Solution: We are given the values $x = -5$, $y = 8$, $x_1 = 3$ and $y_1 = -2$. We must find x_2 and y_2. Substituting into the midpoint formula, we get

$$-5 = \frac{3 + x_2}{2} \qquad \text{and} \qquad 8 = \frac{-2 + y_2}{2}$$

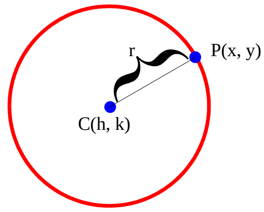
Solving the above for x_2 and y_2, we get:

$$-10 = 3 + x_2 \qquad\qquad 16 = -2 + y_2$$
$$x_2 = -13 \qquad\qquad y_2 = 18.$$

Therefore, the endpoint B has coordinates $(-13, 18)$.

Circles

The set of all points $P(x,y)$ that are a given fixed distance from a given fixed point is called a circle. The fixed distance, r, is called the radius and the fixed point $C(h,k)$ is called the center.



The equation for this set of points can be found by applying the distance formula.
That is,

$$d(C,P) = r.$$

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

If we square both sides of this last equation we get

$$(x-h)^2 + (y-k)^2 = r^2$$

Equation of a circle with $C(h,k)$ and radius r – Standard Form.

$$(x-h)^2 + (y-k)^2 = r^2.$$

Circles with center at the origin, $C(0,0)$ – Standard Form

$$x^2 + y^2 = r^2$$

Example 1: Find the center and radius or the circles a) $x^2 + y^2 = 16$ and b) $(x-3)^2 + (y+2)^2 = 10$.

Solution: a) This circle is in the second form so its center is the origin $\Rightarrow$ $C(0,0)$ and $r^2 = 16 \Rightarrow r = 4$

b) This circle is in the first form so that $h = 3$ and $k = -2$ $\Rightarrow$ $C(3,-2)$ and $r^2 = 10 \Rightarrow r = \sqrt{10}$

Example 2: Write an equation for the circle with $C(3,-5)$ and radius 2.

Solution To write an equation for a specific circle we first write the equation for a circle in standard form:

$$(x-h)^2 + (y-k)^2 = r^2,$$

and then identify the specific values for $h, k,$ and r..

Since we are given the center C and radius r, we can fill in values for h, k, and r as follows:

h = the x-coordinate of $C = 3$; k = the y-coordinate of $C = -5$; and $r = 2$.

The equation of the circle is

Substitute into the equation and simplify: $(x-3)^2 + (y-(-5))^2 = 2^2$

Equation of the circle in standard form: $(x-3)^2 + (y+5)^2 = 4$

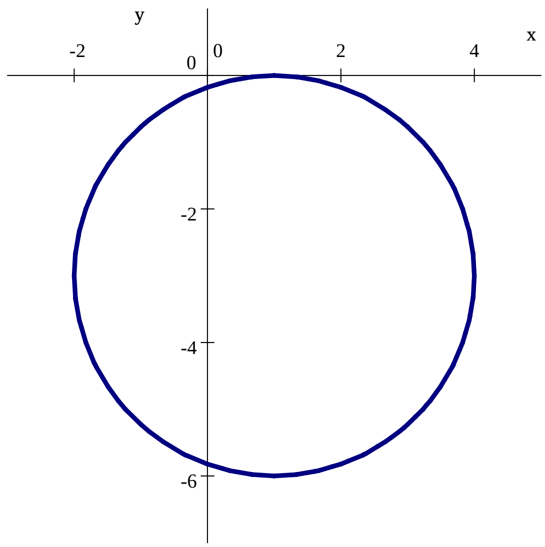
Example 3: Find the center and radius of the circle

$$(x-1)^2 + (y+3)^2 = 9.$$

Graph the circle and find its domain and range.

Solution: From the equation above, we see that $h = 1,\ k = -3$ so the center is $C(1,-3)$. Since $r^2 = 9,$ the radius is $r = 3.$

Graph:



Note that there are no points to the left of the point $(-2,-3)$ nor to the right of $(4,-3)$. Therefore the domain is all real numbers from -2 to $4,$ including -2 and $4,$ or the interval $[-2,4]$. Similarly, the range values include all real numbers between -6 and $0,$ including the endpoints, which is the interval $[-6,0]$.

Note:We can find the domain of the circle algebraically by adding and subtracting the radius 3 to the h value 1.

$$[1-3,\ 1+3] = [-2,4].$$

We can obtain the range by adding and subtracting the radius 3 from the k value -3.

$$[-3-3,\ -3+3] = [-6,0].$$

Example 4: Write an equation for the circle with at the origin and radius $\sqrt{5}$.

Solution: Substituting $h = 0, k = 0,$ and $r = \sqrt{5}$ into the equation for a circle, we get

$$(x-0)^2 + (y-0)^2 = (\sqrt{5})^2$$

$$x^2 + y^2 = 5$$

Since the center of the circle is the origin, we can substitute directly into the formula $x^2 + y^2 = r^2$ to get $x^2 + y^2 = \sqrt{5}^{\,2} \Rightarrow x^2 + y^2 = 5.$

If we expand the equation for the circle $(x-1)^2 + (y+3)^2 = 4,$ we get

$$x^2 - 2x + 1 + y^2 + 6y + 9 = 4.$$

Simplifying and arranging terms, we get

$$x^2 + y^2 - 2x + 6y + 6 = 0$$

We call this the general form of an equation of a circle. Notice that the coefficients of x^2 and y^2 are the same. This is your first clue that a particular equation may be a circle.

The equation $(x-h)^2 + (y-k)^2 = r^2$ form is called the standard form of an equation of a circle.

General Form of an Equation of a Circle

$$x^2 + y^2 + cx + dy + e = 0$$

It is important to be able to recognize that this equation also represents a circle. Note that the equation contains both an x^2 and a y^2 term and that both coeffiients equal 1.

The general form is not as user-friendly as the standard form. We cannot find the center and radius of the circle by simply inspecting the equation as we can with an equation in standard form. To find the center and radius of a circle that is in general form, we must reverse the above process and write the equation in standard form.

For example, $x^2 + y^2 - 4x - 2y - 4 = 0$ is the equation of a circle. (The coefficients of x^2 and y^2 are positive and equal.)
Grouping the x and y terms and moving the constant to the other side of the equation, we get

$$x^2 - 4x + y^2 - 2y = 4$$

We must now complete the square on the x and y terms, and add the calculated amounts to both sides of the equations.

$$x^2 - 4x + \underline{4} + y^2 - 2y + \underline{1} = 4 + \underline{4} + \underline{1}$$

Question: Why did we add 4 to both sides of the equation?
Answer: We added 4 go be able to write $x^2 - 4x$ as a square.

$$\begin{aligned} x^2 - 4x &= x^2 - 4x + 4 - 4 \\ &= (x-2)^2 - 4. \end{aligned}$$

So that we're not changing the equation we must either add 4 and subtract 4 from the same side of the equation $(4 - 4 = 0)$, or we must add 4 to the each side of the equation ($x = y$ is equivalent to $x + 4 = y + 4$).

Writing in factored form, we get the standard form of the equation:

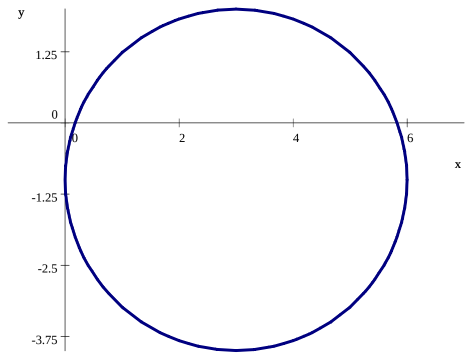
$$(x-2)^2 + (y-1)^2 = 9$$

From this form we can determine that the center of the circle is $(2, 1)$ and the radius is 3. We can use this information to graph the equation by plotting the center $(2, 1)$ and locating as many points on the circle as needed (3 units in any direction from the center) .

Example 5: Find the center and radius of the circle $x^2 + y^2 - 6x + 2y + 1 = 0$. Graph the circle.

Solution: To find the center and radius we must write the equation in standard form:

- Group x and y terms: $x^2 - 6x \quad + y^2 + 2y + 1 = 0$
- Move constant term to other side: $x^2 - 6x \quad + y^2 + 2y \quad = -1$
- Complete the square: $x^2 - 6x \ + \underline{9} \ + y^2 + 2y + \underline{1} = -1 + \underline{9} + \underline{1}$
- Rewrite in factored form: $(x-3)^2 + (y+1)^2 = 9$
- Determine C and r: $C(3,-1) \qquad r = 3$
- Graph by plotting the center $C(3,-1)$ and applying the radius of 3 units to find points on the circle:



Since the radius is 3, the domain is $[3-3,\ 3+3] \ \Rightarrow \ [0,6]$ and the range is $[-1-3,\ -1+3] \ \Rightarrow \ [-4,2]$.

Example 6: Find the center and radius of the circle $3x^2 + 3y^2 - 6x + 12y + 2 = 0$.

Solution: Although the coefficients of x^2 and y^2 are not 1, the equation represents a circle because they are equal, so we divide the equation by the common coeffient.

Divide equation by 3 : $$\frac{\boxed{3}x^2 + \boxed{3}y^2 - 6x + 12y + 2}{\boxed{3}} = \frac{0}{\boxed{3}}$$

$$x^2 + y^2 - 2x + 4y + \frac{2}{3} = 0$$

Group x and y terms: $x^2 - 2x \quad + y^2 + 4y + \frac{2}{3} = 0$

Move constant term to other side: $x^2 - 2x \quad + y^2 + 4y \quad = -\frac{2}{3}$

Complete the square: $x^2 - 2x \ + \underline{1} \ + y^2 + 4y + \underline{4} = -\frac{2}{3} + \underline{1} + \underline{4}$

Rewrite in factored form: $(x-1)^2 + (y+2)^2 = \frac{13}{3}$

Determine C and r: $C(1,-2) \qquad r = \sqrt{\frac{13}{3}} = \frac{\sqrt{39}}{3}$

Exercises for Chapter 3A - Rectangular Coordinate System

1. Plot the following points on a rectangular coordinate system:
a) $A(-2,-1)$ b) $B(3,5)$ c) $C(0,-3)$ d) $D(-4,1)$ e) $E(-1,0)$ f) $F(2,-3)$

2. Use the points in exercise #1 to answer the following questions.
a) Which point(s) are in quadrant I? quadrant II? quadrant III? quadrant IV?
b) Which point(s) are on the *x*-axis? the *y*-axis?
c) Which point(s) meet the condition: $x > 0$?
d) Which point(s) meet the condition: $y \le 0$?
e) Which point(s) meet both the conditions: $x \ge 0$ and $y < 4$?

3. Shade the region of the coordinate plane that contains each of the following sets of points.
a) $\{(x,y)|x \le 0\}$
b) $\{(x,y)|x > -3 \text{ and } y \ge 0\}$
c) $\{(x,y)|x \le -2 \text{ and } y < 2\}$
d) $\{(x,y)|1 < x \le 4 \text{ and } y \ge -1\}$

4. Graph the following relations and write their domain and range.
a) $\{(-7,-2), (4,-3), (0,3), (1,-6), (-2,5), (0,0), (9,-2)\}$
b) $\{(\sqrt{3},-1), (\frac{1}{2},3), (-1,2.5), (-4.3,\sqrt{2}), (-5.6,2.75)\}$

5. Write the domain and range of each of the relations in exercise 3.

6. Find the distance between the following sets of points.
a) $A(-2,-1)$ and $B(3,5)$
b) $C(0,-3)$ and $D(-4,1)$
c) $E(-1,0)$ and $F(2,-3)$
d) $G(\frac{1}{2},\frac{-2}{3})$ and $H(\frac{3}{2},\frac{-5}{3})$
e) $J(-3.3,\ 4.9)$ and $K(1.2,\ -7.5)$

7. Is $P(7,2)$ closer to point $R(-1,3)$ or point $Q(9,-4)$?

8. Prove that the triangle with vertices $A(-3,-2)$, $B(-2,2)$, and $C(6,0)$ is a right triangle.

9. Find the area of triangle *ABC* in exercise #9.

10. Determine whether the triangle with vertices $A(-1,-2)$, $B(2,5)$, and $C(9,2)$ is an isosceles triangle. (An isosceles triangle has two sides that are equal.)

11. If the distance betweeen $A\ (3,x)$ and $B(6,0)$ is 5 units, find all possible coordinates for *A*.

12. Write an equation that describes **all** the points (x,y) that are 5 units from point $B(6,0)$.

13. If the center of a circle is $(2,9)$ and $(0,-5)$ is a point on the circle, find the radius of the circle.

14. Find the midpoint of the line segment connecting the following pairs of points:
a) $A(-2,-1)$ and $B(3,5)$
b) $C(0,-3)$ and $D(-4,1)$
c) $E(-1,0)$ and $F(2,-3)$
d) $G(\frac{1}{2},\frac{2}{3})$ and $H(\frac{3}{2},\frac{-5}{3})$
e) $J(-3.3,\ 4.9)$ and $K(1.2,\ -7.5)$

15. In each of the following, $M(x,y)$ is the mid-point of *A* and *B*. Find the missing end-point, *A* or *B*.
a) $A(-2,5)$; $M(3,2)$
b) $M(1,0)$; $B(-5,-4)$

16. If $(-7,2)$ and $(5,8)$ are endpoints of a diameter of a circle, find the center.

17. Each of the following points A is an endpoint of a line segment. If the midpoint of the line segment AB is the point $(0,0)$, find B.

a) $(-3,5)$ b) $(4,-1)$ c) $(4,2)$ d) $(-5,-6)$ e) $(3,0)$

18. Find the $C(h,k)$ and radius r for the following circles. Then graph the circle.

a) $(x-3)^2+(y-7)^2 = 16$

b) $x^2+y^2 = 49$

c) $(x+\frac{1}{2})^2+(y+5)^2 = 17$

d) $x^2+(y-3)^2 = 20$

e) $x^2+y^2+8x+4 = 0$

f) $2x^2+2y^2-8x+20y-2 = 0$

19. Write an equation for each of the following circles.

a) $C(3,-2)$; $r = \sqrt{13}$

b) $C(0,0)$; $r = 2$

c) $C(-1,5)$ through $(7,-3)$

d) $C(8,-3)$ that touches the y-axis at $(0,-3)$

20. If $(3,7)$ and $(-5,-1)$ are endpoints of a diameter of a circle, write the equation for the circle.

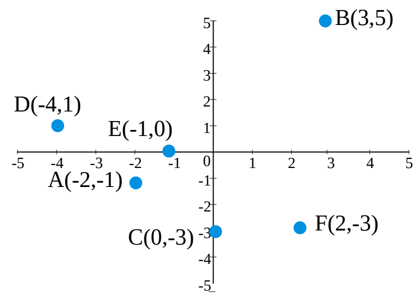
21. Prove that the point $C(2,3)$ is equidistant from $A(3,-2)$ and $B(7,4)$. Is C the midpoint of $\overline{AC}$?

Verify your answer.

22. If the diagonals (line segments connecting opposite vertices) of a parallelogram (a quadrilateral whose opposite sides are equal and parallel) are equal, the parallelogram is a rectangle (a parallelogram with four right angles). If all sides of the rectangle are equal, it is a square. Determine whether the quadrilateral with vertices $A(-1,3)$, $B(-2,7)$, $C(2,8)$, and $D(3,4)$ is a parallelogram, rectangle, or square.

Answers to Exercises for Chapter 3A - Rectangular Coordinate System

1.



2. a) QI: *B*; QII: *D*; QIII: *A*; QIV: *F*

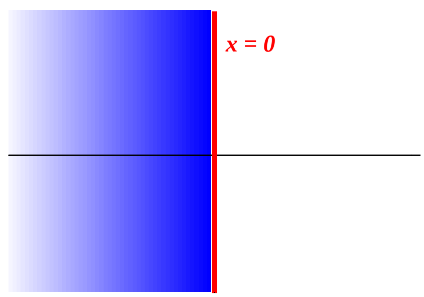
b) *x*-axis: *E*; *y*-axis: *C*

c) *B* and *F*

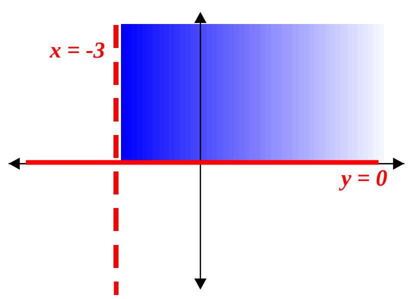
d) *A*, *C*, *E*, and *F*

e) *C* and *F*

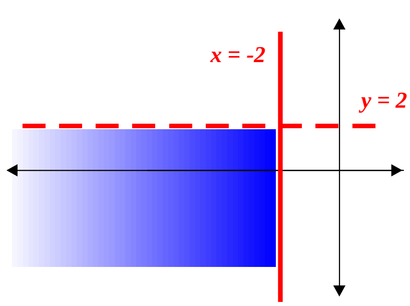
3. a)



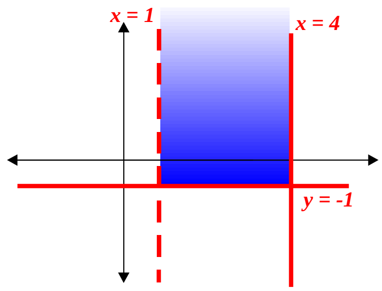
b)



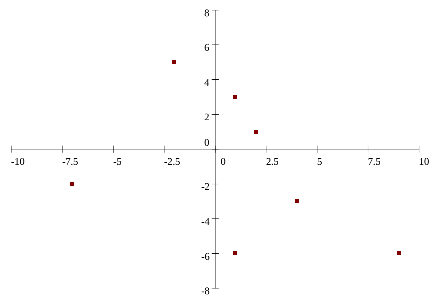
c)



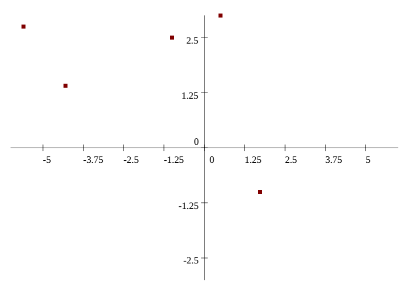
d)



4. a)



Domain: $\{-7, 4, 1, -2, 2, 9\}$
Range: $\{-2, -3, 3, -6, 5, 1\}$
b)



Domain: $\left\{\sqrt{3}, \frac{1}{2}, -1, -4.3, -5.6\right\}$
Range: $\left\{-1, 3, 2/5, \sqrt{2}, 2.75\right\}$

5. a) Domain: all real numbers ≤ 0 Interval notation: $(-\infty, 0]$
Range: all real numbers Interval notation: $(-\infty, +\infty)$
b) Domain: all real numbers greater than -3 Interval notation: $(-3.+\infty)$
Range: all real numbers ≥ 0 Interval notation: $[0, +\infty)$
c) Domain: all real numbers ≤ -2 or $(-\infty, -2]$
Range: all real numbers < 2 or $(-\infty, 2)$
d) Domain: real numbers between 1 and 4, including 4 or $(1, 4]$
Range: real numbers ≥ -1 or $[-1, +\infty)$

6. a) $d(A,B) = \sqrt{(-2-3)^2 + (1-5)^2} =$
$\sqrt{(-5)^2 + (-4)^2} =$
$\sqrt{25+16} =$
$\sqrt{61}$

Note: $\sqrt{61}$ is between $\sqrt{49} = 7$ and $\sqrt{64} = 8$.

b) $d(C,D) = \sqrt{(0-(-4))^2 + (-3-1)^2} =$
$\sqrt{4^2 + (-4)^2} =$

$\sqrt{32} =$

$4\sqrt{2} \approx 5.6569$

c) $d(E,F) = \sqrt{(-1-2)^2 + (0-(-3))^2} = \sqrt{(-3)^2 + (3)^2} = \sqrt{18} = 3\sqrt{2} \approx 4.2426$

d) $d(G,H) = \sqrt{(\frac{1}{2} - \frac{3}{2})^2 + (-\frac{2}{3} - (-\frac{5}{3}))^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2} \approx 1.4142$

e) $d(J,K) = \sqrt{(-3.3-1.2)^2 + (4.9-(-7.5))^2} = \sqrt{174.01} \approx 13.19$

7. It's always helpful to graph the given situation on graph paper to help you visualize the problem. Rememeber what formulas you have to work with and what information each formula provides. To determine whether point R or point Q is closer to P, we must determine the distance from R to P and from Q to P. The smaller of the two distances will tell us which point is closer.

$d(P,R) = \sqrt{65}$

$d(P,Q) = \sqrt{40}$

Since $d(P,Q) < d(P,R)$, Q is closer to P.

8. Plot the three points on graph paper, label them and draw the triangle. Remember that you can prove that a triangle is a right triangle using the Pythagorean Theorem: $a^2 + b^2 = c^2$ where a and b are the sides of the triangle and c is the hypotenuse. What formula we can use to find the length of each side? The distance formula, of course.

$d(A,B) = \sqrt{17}$

$d(B,C) = \sqrt{68} = 2\sqrt{17}$

$d(A,C) = \sqrt{85}$

Once we have the lengths of each side, we plug the smaller sides into the Pythagoream Theorem for a and b, and the largest in for the hypotenuse.

$$\left(\sqrt{17}\right)^2 + \left(\sqrt{68}\right)^2 = \left(\sqrt{85}\right)^2$$

$$17 + 68 = 85$$

$$85 = 85$$

Therefore, ΔABC is a right triangle.

9. To find the area of a triangle we can use the formula $A = \frac{1}{2}bh$, where b is the base and h is the height. Note that h is the perpendicular distance from the third vertex back to the base. Since ΔABC is a right triangle, the two legs are the base and the height , so that

$$A = \tfrac{1}{2}\left(\sqrt{17}\right)\left(2\sqrt{17}\right) = \tfrac{1}{2} \cdot 2\left(\sqrt{17}^2\right) = 17.$$

The area of ΔABC is 17 square units.

10. Remember to plot the points and draw the triangle to help you visualize the problem. To show that ΔABC is an isoscleles triangle we must show that two of the three sides are equal in length. Your plot above will probably show you which sides to try first.

$d(A,B) = \sqrt{58}$

$d(B,C) = \sqrt{58}$

Since sides AB and BC are equal in length ΔABC is an isosceles triangle.

11. Stating the problem algebraically gives us $d(A,B) = 5$. Replacing the left side with the distance formula, we get

$$\sqrt{(3-6)^2 + (x-0)^2} = 5$$

$$\sqrt{(-3)^2 + x^2} = 5$$

$$\sqrt{9 + x^2} = 5$$

We must square both sides of the equation and solve for x.

$$\left(\sqrt{9+x^2}\right)^2 = 5^2$$

$$9 + x^2 = 25$$

$x^2 = 25 - 9$

$x^2 = 16$

$x = \pm 4$

Therefore A can be either of the points $(3,4)$ and $(3,-4)$.

Justify this to yourself by plotting the points on graph paper.

12. $d(P,B) = 5$

$\sqrt{(x-6)^2 + (y-0)^2} = 5$

$(x-6)^2 + y^2 = 25$

Plot point B on graph paper. Sketch out the points that are 5 units from point B. What shape do you get? A circle.

13. Plot the center and point and sketch the circle. The radius of a circle r is the distance from the center of the circle to any point on the circle. Using the distance formula, we get

$r = \sqrt{(2-0)^2 + (9-(-5))^2} = \sqrt{200} = 10\sqrt{2}$

14. Find the midpoint of the line segment connecting the following pairs of points:

a) Using the midpoint formula gives us $M\left(\frac{-2+3}{2}, \frac{-1+5}{2}\right) = M\left(\frac{1}{2}, 2\right)$

b) $M\left(\frac{0+-4}{2}, \frac{-3+1}{2}\right) = M(-2,-1)$

c) $M(\frac{-1+2}{2}, \frac{1+(-3)}{2}) = M\left(\frac{1}{2}, \frac{-3}{2}\right)$

d) $M\left(\frac{\frac{1}{2}+\frac{3}{2}}{2}, \frac{\frac{2}{3}+\left(-\frac{5}{3}\right)}{2}\right) = M(1, -\frac{1}{2})$

e) $M\left(\frac{-3.3+1.2}{2}, \frac{4.9+(-7.5)}{2}\right) = \left(\frac{-2.1}{2}, \frac{-2.6}{2}\right) = M(-1.05, -1.3)$

15. a) Use Example 2 in the midpoint notes as a model for this problem. You should get the point $(8,-1)$ as your answer.

b) $(7,4)$

16. Plot the given points on graph paper, draw the circle throught the two points, and draw the diameter. The diameter of a circle will always go through its center. Moreover, the center will be the mid-point of the diameter. To find the center, we must then find the midpoint of the two endpoints of the diameter using the midpoint formula.

$Center: \left(\frac{-7+5}{2}, \frac{2+8}{2}\right) = (-1,5)$

17. a) $(3,-5)$ b) $(-4,1)$ c) $(-4,-2)$ d) $(5,6)$ e) $(-3,0)$

Note that the signs of the coordinates are opposites. If the origin is the midpoint of two points, we say that the two points have symmetry with respect to the origin.

18. a) $(x-3)^2 + (y-7)^2 = 16$ Recall the formulas for circles. $C(3,7)$; $r = 4$

b) $C(0,0)$; $r = 7$

c) $C(-\frac{1}{2}, -5)$; $r = \sqrt{17}$

d) $C(0,3)$; $r = \sqrt{20} = 2\sqrt{5}$

e) Group x-terms and move 4 to other side of the equation: $x^2 + 8x \quad + y^2 = -4$ Complete the square on the x-terms by adding $[\frac{1}{2} \cdot 8]^2 = 16$ to both sides of the equation.

$x^2 + 8x + 16 + y^2 = -4 + 16$

Simplify: $(x+4)^2 + y^2 = 12$

Therefore, $C(-4,0)$; $r = \sqrt{12} = 2\sqrt{3}$

f) $x^2 - 4x \quad + y^2 + 10y \quad = 1$

$x^2 - 4x + 4 + y^2 + 10y + 25 = 1 + 4 + 25$

$(x-2)^2 + (y+5)^2 = 30$

$C(2,-5)$; $r = \sqrt{30}$

19. a) $(x-3)^2 + (y+2)^2 = 13$

b) $x^2 + y^2 = 4$

c) We are given the center of the circle $(-1,5)$ so $h = -1$ and $k = 5$, but we do not have the radius. Plot the center and draw the circle through P. What do we need to do to find the length of the radius? The distance formula !

$r = d(C,P) = \sqrt{(-1-7)^2 + (5-(-3))^2} = \sqrt{128}$

Substituting into the formula, we get

$(x+1)^2 + (y-5)^2 = \left(\sqrt{128}\right)^2$

$(x+1)^2 + (y-5)^2 = 128$

d) Plot the points on graph paper. You should be able to see that the radius of the circle is 8.

Therefore the equation is $(x-8)^2 + (y+3)^2 = 64$

20. The diameter of a circle goes through the center of a circle. Therefore, you can use the midpoint formula to find the center of the circle. $C(-1,3)$ The radius is the distance from the center to either of the points on the circle. $r = \sqrt{32}$ $(x+1)^2 + (y-3)^2 = 32$

21. Plot out the problem on graph paper. Use the distance and midpoint formulas appropriately to draw your conclusions algebraically. You should find that C is equidistant from A and B, but it is not the midpoint. Explain to yourself why.

Draw all the points that are equidistant from A and B. You should have drawn a line. This line is called the perpendicular bisector of line segment AB.

22. Using the distance formula appropriately should show that all sides are equal in length and the diagonals are equal, therefore the most accurate term for the quadrilateral is square. Note that a square is also a parallelogram and a rectangle.

Chapter 3B - Graphs of Equations

Graphing by Plotting Points

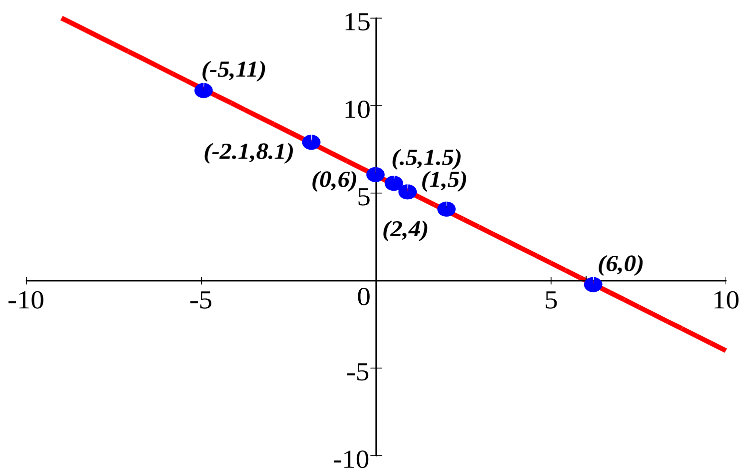
We have seen that the coordinate system provides a method for locating points in a plane. Furthermore, we can plot sets of ordered pairs on the coordinate system to visualize the relationship between the two variables. However,in most cases we will be interested in relations that are stated as equations. For example, the equation $x+y=6$. refers to the relation $\{(x,y)|x+y=6\}$, read "the set of all pairs (x,y) such that $x+y=6$". Every pair of numbers (x,y) that makes the equation true is called a solution to the equation. The pair $(2,4)$ is a solution to the equation $x+y=6$ because $2+4=6$; but the ordered pair $(4,4)$ is not a solution since $4+4\neq 6$.

Question: How many solutions does the equation $x+y=6$ have?
Answer: There are an infinite number of solutions.

Example 1: Find three more pairs (x,y) such that $x+y=6$.
Solution: Any pair of numbers whose sum is 6 is a solution. For example:

$$(-5,11),\ (-2.1,8.1),\ (1,5),\ (0,6),\ (6,0),\ (\tfrac{1}{2},5\tfrac{1}{2}),\ (2,4)$$

The plot of every point that corresponds to a solution to the equation $x+y=6$ is the graph of the equation $x+y=6$:



The points labeled above are the solutions we listed in example 1; however, the graph consists of **every** pair (x,y) that is a solution to the equation. Furthermore, every point on the graph satisfies the equation $x+y=6$..

Note that the above graph is a line. The graph of any linear equation is a line. A linear equation is one that can be written in the form $ax+by-c=0$, where a and b are not both 0. The equation $x+y=6$ can be written in the form of a linear equation $x+y-6=0$, where $a=1$, $b=1$, and $c=-6$.

IMPORTANT CONCEPT: **The graph of an equation consists of all pairs (x,y) that are solutions to the equation. Every solution to the equation is a point on the graph and every point on the graph (x,y) is a solution to the equation.**

Example 2: Which of the following pairs are points on the graph of $2x + y = 10$?

a) $(-4, 18)$
b) $(5, 1)$
c) $(4, -2)$
d) $(0, 10)$
e) $\left(\frac{1}{2}, 9\right)$

Solution:

a) $(-4, 18)$ is a solution because

$$2(-4) + 18 = 10$$
$$-8 + 18 = 10$$
$$10 = 10 \text{ is a true statement.}$$

Therefore$(-4, 18)$ is a point on the graph.

b) $(5, 1)$ is not a solution because

$$2(5) + 1 = 10$$
$$10 + 1 = 10$$
$$11 \neq 10$$

Therefore, $(5, 1)$ is not on the graph.

c) $(4, -2)$ is not a solution because

$$2(4) + (-2) = 10$$
$$6 \neq 10$$

Therefore, $(4, -2)$ is not on the graph.

d) $(0, 10)$ is a solution

$$2(0) + 10 = 10$$
$$0 + 10 = 10$$
$$10 = 10 \text{ is a true statement.}$$

Therefore, $(0, 10)$ is a point on the graph.

e) $(\frac{1}{2}, 9)$ is a solution.

$$2(\tfrac{1}{2}) + 9 = 10$$
$$1 + 9 = 10$$
$$10 = 10$$

Therefore, $(\frac{1}{2}, 9)$ is a point on the graph.

To find a solution to an equation, choose a value for x, substitute it into the equation and solve for y.

Example 3: Find three solutions to the equation $xy = 10$.

Solution:

If $x = 1$, $(1)y = 10 \Rightarrow y = 10 \Rightarrow (1, 10)$ is a solution.

If $x = 2$, $(2)y = 10 \Rightarrow y = 5 \Rightarrow (2, 5)$ is a solution.

If $x = -5$, $(-5)y = 10 \Rightarrow y = -2 \Rightarrow (-5, -2)$ is a solution.

We can write this in chart form:

x	y	(x, y)
1	10	$(1, 10)$
2	5	$(2, 5)$
-5	-2	$(-5, -2)$

Note that each of the pairs (x, y) will be coordinates of points on the graph of $xy = 10$.

Question: Is the point $(0,0)$ on the graph of $xy = 10$?

Answer: $(0,0)$ is not a point on the graph. For if it were, then the following equations would be true

$$(0)(0) = 10$$
$$0 = 10$$

In fact, for the same reason, any point of the form $(0,y)$ is not on the graph of the equation $xy = 10$.

Example 4: Find fifteen solutions to the equation $x^2 + y^2 = 25$.

Solution: Since we are going to substitute values for x and solve for y, it will be easier to first solve for y:

$$x^2 + y^2 = 25$$
$$y^2 = 25 - x^2$$
$$y = \pm\sqrt{25 - x^2}$$

Substitute values for x into $y = \pm\sqrt{25 - x^2}$, and solve for y.

x	y	(x,y)
-6	$\pm\sqrt{25-(-6)^2} = \pm\sqrt{-11} = -$	$-$
-5	$\pm\sqrt{25-(-5)^2} = 0$	$(-5,0)$
-4	$\pm\sqrt{25-(-4)^2} = \pm 3$	$(-4,-3)$
		$(-4,3)$
-3	$\pm\sqrt{25-(-3)^2} = \pm 4$	$(-3,-4)$
		$(-3,4)$
-2	$\pm\sqrt{25-(-2)^2} = \pm\sqrt{21}$	$\left(-2,-\sqrt{21}\right)$
		$\left(-2,\sqrt{21}\right)$
0	$\pm\sqrt{25-(0)^2} = \pm 5$	$(0,-5)$

x	y	(x,y)
1	$\pm\sqrt{25-(1)^2} = \pm\sqrt{24}$	$\left(1,-\sqrt{24}\right)$
		$\left(1,\sqrt{24}\right)$
2	$\pm\sqrt{25-(2)^2} = \pm\sqrt{21}$	$\left(2,-\sqrt{21}\right)$
		$\left(2,\sqrt{21}\right)$
3	$\pm\sqrt{25-(3)^2} = \pm 4$	$(3,-4)$
		$(3,4)$
4	$\pm\sqrt{25-(4)^2} = \pm 3$	$(4,-3)$
		$(4,3)$
5	$\pm\sqrt{25-(5)^2} = 0$	$(5,0)$

You should recognize that this equation gives us a graph that is a circle, and each of the points we found above is a point on the circle.

Note that any values for x that are less than -5 or greater than 5 will give us negative values under the radical sign. Therefore, x-coordinates of the solutions will be in the interval $[-5,5]$, which is the domain of the relation.

To graph an equation by plotting points:

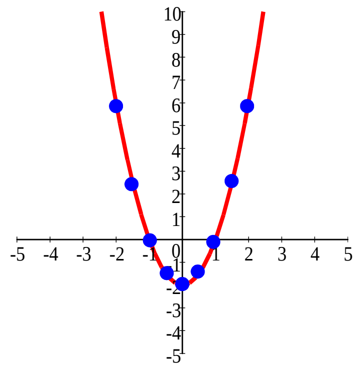
1. Solve the equation for y.
2. Complete a table of values by substituting your choice of values for x into the equation, then solving for y. Use as many points as necessary to determine the shape of the graph.
3. Plot the points and draw a smooth curve through them..

Example 5: Graph the equation $4x^2 - 2y = 4$.

Solution:

- Solve for y: $-2y = -4x^2 + 4 \Rightarrow y = 2x^2 - 2$
- Complete a table of values by picking an x, and then using the above equation to determine the corresponding y. Choose enough points to determine the shape of the graph.
- Plot the points and draw the curve.

x	$y = 2x^2 - 2$	(x,y)
-2	6	$(-2,6)$
-1.5	2.5	$(-1.5,2.5)$
-1	0	$(-1,0)$
-0.5	-1.5	$(-.5,-1.5)$
0	-2	$(0,-2)$
0.5	-1.5	$(.5,-1.5)$
1	0	$(1,0)$
1.5	2.5	$(1.5,2.5)$
2	6	$(2,6)$



Note that the points on this graph do not form a straight line. This is because the equation $y = 2x^2 - 2$ is not in the form of a linear equation because x is raised to the second power. Therefore, the graph will not be a line. This graph is called a parabola. The graph of a quadratic equation is a parabola. A quadratic equation is an equation that can be written in the form $y = ax^2 + bx + c$, where $a \neq 0$. The equation $y = 2x^2 - 2$ is a quadratic, where $a = 2$, $b = 0$, $c = -2$.

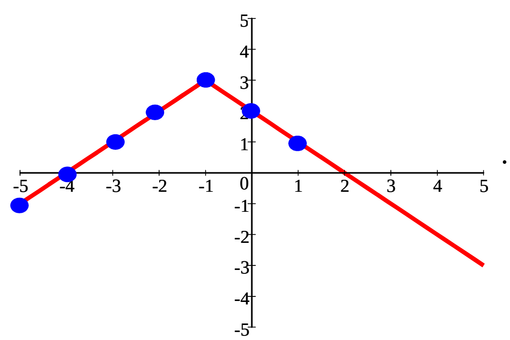
Note that any value for x will give a real value for y, so that the domain of the relation is $(-\infty, +\infty)$. The y-values include all real numbers greater than or equal to -2, so that the range is $[-2, +\infty)$.

Example 6: Graph the equation $y = -|x+1| + 3$.

Solution:

- Solve for y: $y = -|x+1| + 3$.
- Complete a table of values.
- Plot the points and draw the curve.

x	$y = -\|x+1\| + 3$	(x,y)
-5	-1	$(-5,-1)$
-4	0	$(-4,0)$
-3	1	$(-3,1)$
-2	2	$(-2,2)$
-1	3	$(-1,3)$
0	2	$(0,2)$



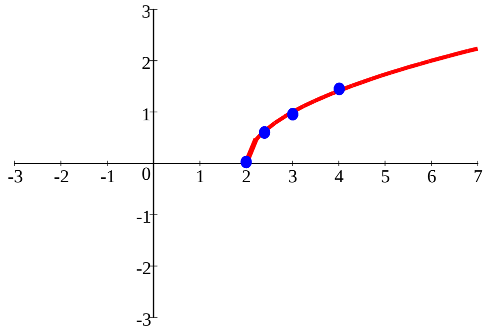
Note that the graph continues infinitely to the left and to the right, because we could place any number in the equation for x and get a corresponding value for y. Therefore, the domain of the relation is $(-\infty, +\infty)$. From the graph you should be able to see that there are no y-values larger than 3, so that the range is $(-\infty, 3]$.

Example 7: Graph the equation $y = \sqrt{x-2}$.

Solution:

- Solve for y: $y = \sqrt{x-2}$.
- Complete a table of values.
- Plot the points and draw the curve.

x	$y = \sqrt{x-2}$	(x,y)
0	–	no solution
1	–	no solution
2	0	$(2,0)$
2.5	$\sqrt{0.5}$	$(2.5, \sqrt{0.5}^{*})$
3	1	$(3,1)$
4	2	$(4,2)$



$^{*}\sqrt{0.5} \approx 0.7071.$

Note that the graph continues infinitely to the right because we could place any number in the equation for x that is greater than 2 and get a corresponding value for y.

Question: What happens to the graph when $x < 2$?

Answer: When numbers were substituted for x that were less than 2, we could not compute y because we had a negative value under the radical. Therefore the graph does not extend to the left of the point $(2,0)$.

Since x must be greater than or equal to 2 for y to be a real number, the domain of the relation is $[2,+\infty)$. By examining the graph and table of values, note that the y-coordinates are greater than or equal to 0, so that the range is $[0,+\infty)$.

Intercepts

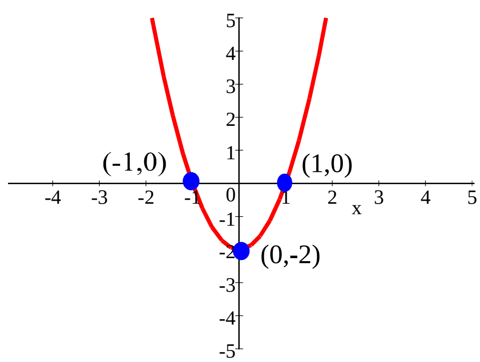
Points that will prove to be very important to our future study of equations and their graphs are the intercepts.

Definition: The points where the graph of an equation crosses the *x*-axis is called the *x*-**intercept**. If $(a, 0)$ is a point on the graph of an equation, we say that *a* is an *x*-intercept of the graph because $(a, 0)$ is a point on the *x*-axis. Note that the *x*-intercept will always have a *y*-coordinate of 0. The *x*-intercepts are also called zeros of the equation. A zero is a value for *x* that causes the corresponding *y*-value to be 0. If the point $(-1, 0)$ is an *x*-intercept for the graph of an equation, -1 is a **zero** of the equation.

Definition: The points where the graph of an equation crosses the *y*-axis is called the *y*-**intercept**. If $(0, b)$ is a point on the graph of an equation, we say that *b* is a *y*-intercept of the graph because $(0, b)$ is a point on the *y*-axis. Note that the *y*-intercept will always have an *x*-coordinate of 0.

By inspecting the table of values and the graph of $y = 2x^2 - 2$, we see the equation crosses the *y*-axis at the point $(0, -2)$ and the *x*-axis at the points $(-1, 0)$ and $(1, 0)$. Therefore, -2 is the *y*-intercept and 1 and -1 are *x*-intercepts.

x	y
-2	6
-1.5	2.5
-1	0
-0.5	-1.5
0	-2
0.5	-1.5
1	0
1.5	2.5
2	6



Finding the *y*-intercepts

To find the *y*-intercept(s) of an equation, substitute 0 for *x*, and solve for *y*.

Finding the *x*-intercepts

To find the *x*-intercept(s) of an equation, substitute 0 for *y*, and solve for *x*.

Example 1: Find the *x*- and *y*-intercepts of $y = 2x^2 - 2$ algebraically.

Solution: Find the intercepts by substituting 0 for the appropriate variable.

- *y*-intercept: Substitute 0 for *x* and solve for *y*: $y = 2(0)^2 - 2 = -2$
  -2 is the *y*-intercept, and $(0, -2)$ is a point on the graph.
- *x*-intercept: Substitute 0 for *y* and solve for *x*: $0 = 2x^2 - 2 \Rightarrow$
  $0 = 2(x^2 - 1) \Rightarrow 0 = 2(x + 1)(x - 1) \Rightarrow x = -1$ and $x = 1$
  -1 and 1 are *x*-intercepts and $(-1, 0)$ and $(1, 0)$ are points on the graph.

When graphing by plotting points, it is important to include points close to and on either side of the x-intercepts.

Question: Why is the above true?

Answer: The x-intercepts are the zeros–that is, they are the values for x that cause $y = 0$.

- The y-values may change sign on either side of a zero which means that a graph may be below the y-axis on one side of an x-intercept and above it on the other, or vice-versa.
- Numbers close to zero often behave differently. For example, if $0 < x < 1$, $x^2 < x$. Prove this to yourself using several fractions that are between 0 and 1. Ex: $\left(\frac{1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2}$

Example 2: Find the intercepts of the graph of $x^2 + y^2 = 25$.

Solution:

- y-intercepts: Substitute 0 for x and solve for y:

$$(0)^2 + y^2 = 25 \quad \Rightarrow y^2 = 25 \quad \Rightarrow y = \pm 5$$

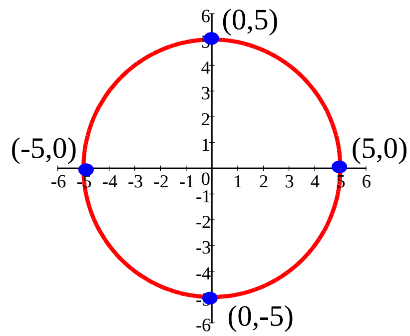
Thus, -5 and 5 are y-intercepts; $(0,-5)$ and $(0,5)$ are points on the graph.

- x-intercepts: Substitute 0 for y and solve for x:

$$x^2 + (0)^2 = 25 \quad \Rightarrow x^2 = 25 \quad \Rightarrow x = \pm 5$$

Thus, -5 and 5 are x-intercepts; $(-5,0)$ and $(5,0)$ are points on the graph.

We can confirm our findings by inspecting the graph of $x^2 + y^2 = 25$ which we know to be a circle with $C(0,0)$ and radius 5.



Example 3: Find the intercepts of $y = x^2 - 5x - 6$.

Solution:

- y-intercepts: Substitute 0 for x and solve for y: $(0,\ y)$

$$y = (0)^2 - 5(0) - 6$$
$$y = -6$$

Thus, the point $(0,-6)$ is the y-intercept.

- x-intercepts: Substitute 0 for y and solve for x: $(x,\ 0)$

$$0 = x^2 - 5x - 6$$
$$0 = (x-6)(x+1)$$
$$x - 6 = 0 \quad \text{and} \quad x + 1 = 0$$
$$x = 6 \quad \text{and} \quad x = -1$$

Thus, the two points $(6,0)$ and $(-1,0)$ are the x-intercepts.

Example 4: Find the intercepts of $xy + y^3 = x^2 - 6x + 8.$

Solution:

- y-intercepts: Substitute 0 for x and solve for y: $\left(0,\ y\right)$

$$(0)y + y^3 = (0)^2 - 6(0) + 8$$
$$y^3 0 = 8$$
$$y = 8^{1/3} = 2$$

Thus, the y-intercept is the point $(0, 2)$.

- x-intercepts: Substitute 0 for y and solve for x: $(x,\ 0)$

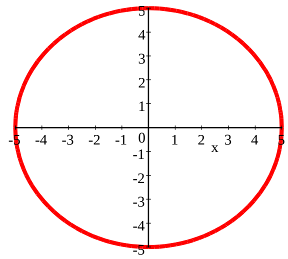
$$x(0) + (0)^3 = x^2 - 6x + 8$$
$$0 = x^2 - 6x + 8$$
$$0 = (x-2)(x-4)$$
$$x - 2 = 0 \quad \text{and} \quad x - 4 = 0$$
$$x = 2 \quad \text{and} \quad x = 4$$

Thus, the x-intercepts are the points $(2, 0)$ and (4.0).

Symmetry

When we think of symmetry in art, we think of balance on both sides of a center of focus. In other situations, the term conjures up the idea of a "mirror image." In either case, the notion of symmetry means essentially the same thing in an algebra class: a shape looks the same on both sides of a dividing line or point. The dividing line is called the line of symmetry. We will also consider points of symmetry in this section.

Examine the graph of the circle $x^2 + y^2 = 25$ for symmetry.

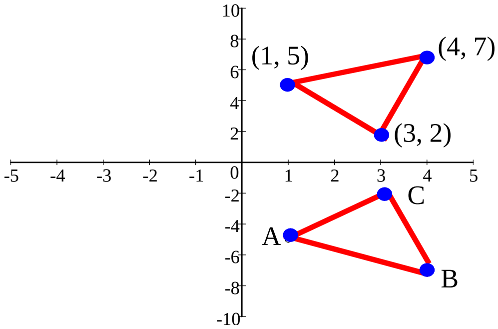


Question: How many lines or points of symmetry can you find for the circle?

Answer: The circle has symmetry about its center–a point, and about any straight line that goes through its center–an infinite number of lines.

Although any line that goes through the center of the circle can be a line of symmetry for the circle, we will focus our efforts in this section on the *x*- and *y*-axes. We will also look at symmetry about the origin.

Example 1: The figures on the axes below are symmetric about the *x*-axis. Find the coordinates of *A*, *B*, and *C*.



Solution: $A(1,-5)$ $B(4,-7)$ $C(3,-2)$

Note that the *x*-coordinates are the same as the corresponding point in the Quadrant I figure, but the *y*-coordinates are the negatives of the *y*-coordinates in the Quadrant I figure.

Example 2: Find the points of the figure that would be symmetric, about the *y*-axis, to the above figure in Quadrant I.

Solution: Corresponding points would be $(-1,5)$ $(-4,7)$ $(-3,2)$.

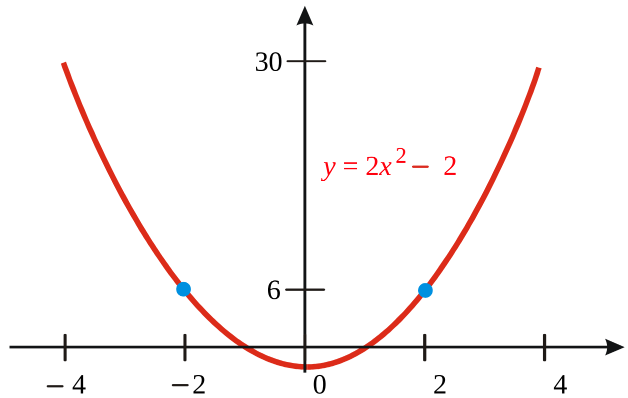
Note that the *y*-coordinates of corresponding points are the same as the Quadrant I figure, but the *x*-coordinates are the negatives.

Symmetric about the *y*-axis.

When we say that a graph is symmetric with respect to the *y*-axis we mean that if the graph is reflected through or reflected about the *y*-axis we get the same graph.

Note that the graph of the equation $y = 2x^2 - 2$ is symmetric about the *y*-axis. That is, if we reflected the graph 180° about the *y*-axis, the graph would be the same. See the plot below.
Examine the table of values that we used to graph $y = 2x^2 - 2$.

x	y
-2	6
-1.5	2.5
-1	0
-0.5	-1.5
0	-2
0.5	-1.5
1	0
1.5	2.5
2	6



Question: What do you notice about the *y*-values when $x = -2$ and $x = 2$? What about the *y*–values for $x = -1.5$ and $x = 1.5$?
Answer: For any number x and its opposite $-x$, the *y*-values are equal.

Definition: A graph is symmetric about the *y*-axis if and only if for every point (x, y) on the graph, $(-x, y)$ will also be a point on the graph.
This definition leads us to the following test for symmetry about the *y*-axis.

Test for Symmetry about the *y*-axis.

To determine algebraically if a graph will be symmetric about the *y*-axis, substitute $-x$ for x and simplify. If the resulting equation is equivalent to the original equation, the graph will be symmetric about the *y*-axis.

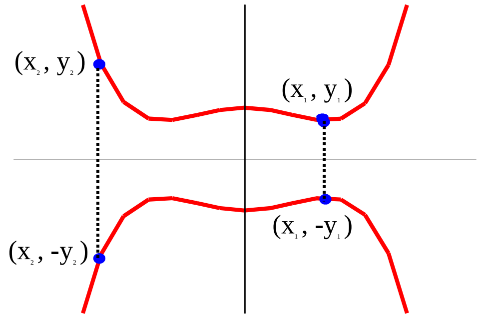
Example 3: Show that $4x^2 - 2y = 4$ has symmetry with respect to the *y*-axis.
Solution: Substituting $-x$ for x in the original equation $4x^2 - 2y = 4$,
we get $4(-x)^2 - 2y = 4$.
Simplifying, $4x^2 - 2y = 4$.
The result is equivalent to the original equation; therefore, the graph of $4x^2 - 2y = 4$ is symmetric about the *y*- axis.

Example 4: Test the equation $x^2y - 5y^3 = x^4$ for symmetry about the *y*-axis.
Solution: Substitute $-x$ for x amd simplify

$$(-x)^2y - 5y^3 = (-x)^4 \quad \Rightarrow \quad x^2y - 5y^3 = x^4$$

Since the resulting equation is the same as the original, the graph is symmetric about the *y*-axis.

The next type of symmetry we discuss is that of a graph being symmetric with respect to the x-axis. The following graph has symmetry about the x-axis. That is, the graph will be the same if it is reflected 180° about the x-axis. For this to occur, every x must be paired with both y and $-y$.



The graph will be the same if it is reflected 180° about the x-axis.

Definition: A graph has symmetry about the x-axis if and only if $(x,-y)$ is a point on the graph for every point (x,y) on the graph.

Test for Symmetry about the *x*-axis:

Substitute $-y$ for y in the equation and simplify. If the resulting equation is equivalent to the original, the graph will be symmetric about the x-axis.

Example 5: Test $x^2y - 5y^3 = x^4$ for symmetry about the x-axis.

Solution: Substitute $-y$ for y and simplify

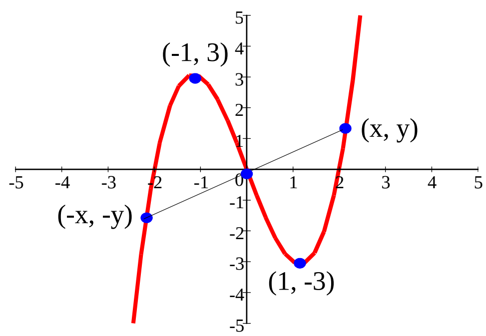
$$x^2(-y) - 5(-y)^3 = x^4 \quad \Rightarrow \quad -x^2y + 5y^3 = x^4$$

Multiply both sides by -1 : $-1(-x^2y + 5y^3) = -1(x^4)$

$$x^2y - 5y^3 = -x^4$$

Since neither of the resulting equations is equivalent to the original, the graph of the equation is not symmetric about the x-axis.

The following graph has symmetry about the origin. That is, it will be the same if it is rotated 180° about the origin. Note that for every point (x,y), the point $(-x,-y)$ will also be on the graph. Furthermore, the line segment connecting these corresponding pairs will go through the origin.



Definition: The graph of an equation will have symmetry about the origin if and only if for every point (x,y) on the graph, the point $(-x,-y)$ will also be on the graph.

Test for Symmetry about the origin:

Substitute $-x$ for x, and $-y$ for y into the equation. If the resulting equation is equivalent to the original, the graph has symmetry about the origin.

Example 6: Test $x^2y - 5y^3 = x^4$ for symmetry about the origin.

Solution: Substitute $-x$ for x, and $-y$ for y, and simplify.

$(-x)^2(-y) - 5(-y)^3 = (-x)^4 \quad \Rightarrow \quad -x^2y + 5y^3 = x^4$

Since the resulting equation is not equivalent to the original equation, there is no symmetry about the origin.

Note that even if we multiply the resulting equation by -1, the result $x^2y - 5y^3 = -x^4$ is not equivalent to the original equation. The results of the examples in this section have shown that $x^2y - 5y^3 = x^4$ has symmetry about the y-axis only.

Example 7: Test the equation $xy^3 - 5x^3y = xy$ for symmetry about the x-axis, y-axis, and origin.

Solution:

- y-axis: [Substitute $-x$ for x]:

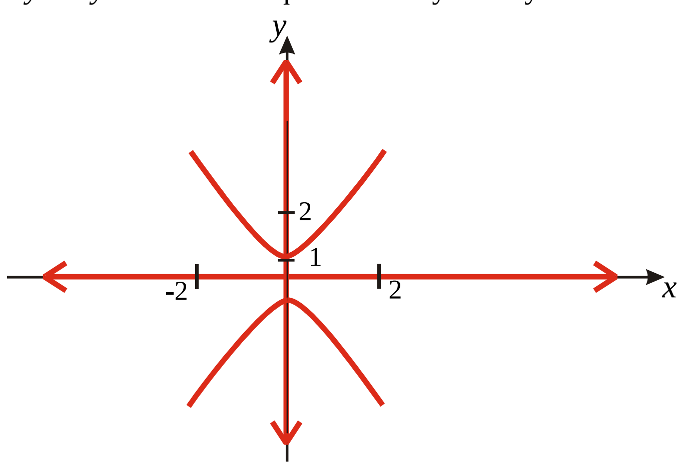
$$(-x)(y)^3 - 5(-x)^3y = (-x)y$$
$$-xy^3 + 5x^3y = -xy$$
$$-(xy^3 - 5x^3y) = -(xy)$$
$$xy^3 - 5x^3y = xy$$

- Since the resulting equation is equivalent to the original, the graph has symmetry about the y-axis.
- x-axis: [Substitute $-y$ for y]:

$$.x(-y)^3 - 5x^3(-y) = x(-y)$$
$$-xy^3 + 5x^3y = -xy$$
$$-(xy^3 - 5x^3y) = -(xy)$$
$$xy^3 - 5x^3y = xy \quad \Rightarrow \text{ equation has symmetry about the } x\text{-axis.}$$

- origin: [Substitute $-x$ for x and $-y$ for y]:

$$(-x)(-y)^3 - 5(-x)^3(-y) = (-x)(-y)$$
$$xy^3 - 5x^3y = xy \quad \Rightarrow \text{ equation has symmetry about the origin.}$$



Example 8: Test the equation $y = x|x^4 - 3|$ for symmetry about the x-axis, y-axis, and origin.

Solution:

- y-axis: [Substitute $-x$ for x]:

  $y = (-x)\left|(-x)^4 - 3\right|$

  $y = -x|x^4 - 3|$ $\Rightarrow$ equation does not have symmetry about the y-axis.

- x-axis: [Substitute $-y$ for y]:

  $(-y) = x|x^4 - 3|$

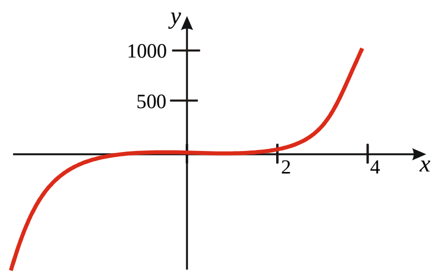
  $-y = x|x^4 - 3|$ $\Rightarrow$ equation does not have symmetry about the x-axis.

- origin: [Substitute $-x$ for x and $-y$ for y]:

  $(-y) = (-x)\left|(-x)^4 - 3\right|$

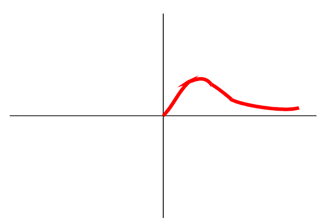
  $-y = -x|x^4 - 3|$

  $y = x|x^4 - 3|$ $\Rightarrow$equation has symmetry about the origin.



Knowing whether the graph of a particular equation has symmetry about a point or line can assist us if we are graphing by plotting points. If we are using a grapher it provides us information about what our graph should look like so that we can determine whether a particular graph is reasonable for the equation we entered.

Consider the section of graph in the figure below.



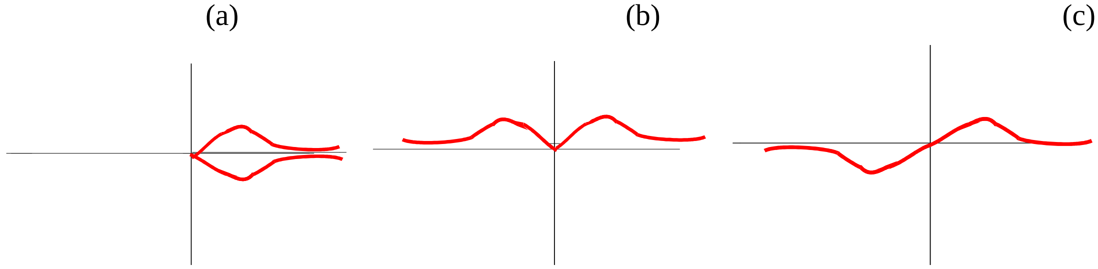
If we knew that the graph had symmetry about the x-axis, y-axis, or origin, we could complete the graph knowing only what the Quadrant I section of the graph looked like.

Example 9: Reproduce the graph above using paper and pencil. Complete the graph for each of the following situations. The graph has symmetry about the

a) x-axis b) y-axis c)origin.

Solution:

(a) (b) (c)



To use symmetry in graphing an equation,

1. Determine whether the graph has symmetry about the x-axis, y-axis, or origin.
2. Plot enough points in the first quadrant to get an idea of what the graph looks like.
3. Apply symmetry to complete the graph.

Example 10: Use symmetry in graphing the equation $xy = 2$.

Solution: Determine symmetry.

x-axis: $x(-y) = 2$

$-xy = 2 \Rightarrow$ No symmetry about x-axis.

y-axis: $(-x)y = 2$

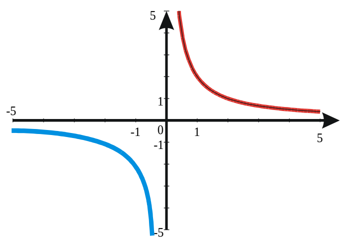
$-xy = 2 \Rightarrow$ No symmetry about y-axis.

origin: $(-x)(-y) = 2$

$xy = 2 \Rightarrow$ Symmetry about the origin.

Because $xy = 2$ has symmetry about the origin, use values of x that are positive in the table of values, and use symmetry to find the remainder of the graph.

x	y
1/2	4
1	2
2	1
4	1/2



Since, for example, the point $(1/2, 4)$ is on the graph of $xy = 2$, and this graph is symmetric about the origin, we know that the point $(-1/2, -4)$ is also a point on the graph of $xy = 2$.

Exercises for Chapter 3B - Graphs of Equations

1. Graph $y = 3x - 2$
2. Graph $6x^2 - 2y = 4x + 2$.
3. Graph $y = |x - 2|$.
4. Graph $y = x^2 + 3$.
5. Graph $y = -\sqrt{x}$.
6. Graph $y = -x^3$.
7. Graph $xy = 1$.
8. Find the intercepts for $3x - 2y = 6$.
9. Find the intercepts for $y = x^2 - 5x + 6$.
10. Find the intercepts for $|x| - y = 2$.
11. Find the intercepts for $xy - 2y = 1$.
12. Find the intercepts for $x^2 + 4x - 2xy = y + 5$.
13. Find the intercepts for $y = \frac{1}{x}$.
14. Test each equation for symmetry about the x-axis, y-axis, or origin.

a) $2x - y = 0$
b) $2x^2 - y = 0$
c) $y = x^3 - 9x$.
d) $x^2 + y^2 - 6x = 2$
e) $xy = x|x^2 - 1|$
f) $xy^2 = x^3|xy|$

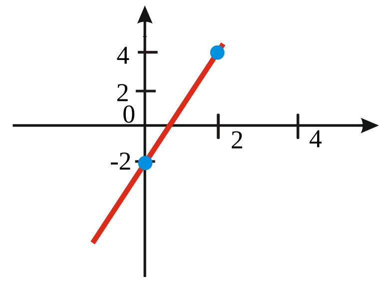
15. Use symmetry about the x-axis, y-axis, or origin to sketch the graph of each.

a) $y = -x^2 - 2$
b) $y = x^4 - 4x^2$
c) $x^2 + 4y^2 = 4$
d) $4x = x^3 - y$

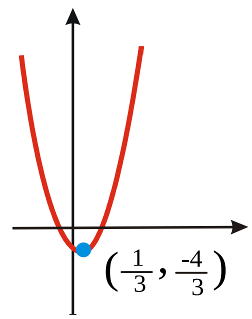
16. In one of the previous examples we plotted some points on the graph of $x^2y - 5y^3 = x^4$, and did not plot any points for which y was positive. Find values of x which satisfy this equation for $y = 1/20$. Hint this equation is a quadratic in x^2.

Answers to Exercises for Chapter 3B - Graphs of Equations

1.



2.



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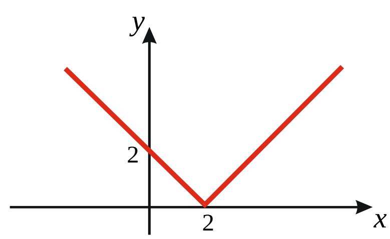

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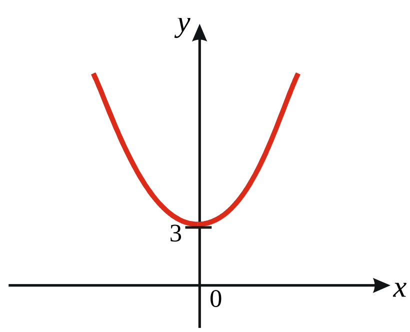

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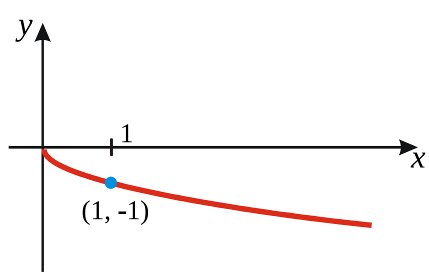

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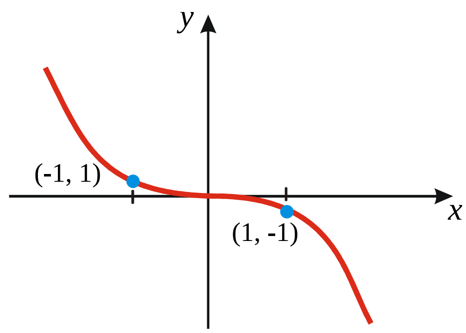

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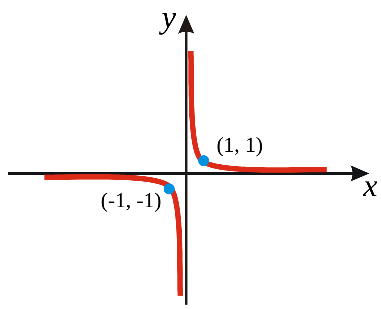


8. y-intercept: Substitute $x = 0$: $3(0) - 2y = 6 \Rightarrow -2y = 6 \Rightarrow y = -3$ or $(0, -3)$
x-intercept: Substitute $y = 0$: $3x - 2(0) = 6 \Rightarrow 3x = 6 \Rightarrow x = 2$ or $(2, 0)$

9. y-intercept: Substitute $x = 0$: $y = (0)^2 - 5(0) + 6 \Rightarrow y = 6$ or $(0, 6)$
x-intercept: Substitute $y = 0$: $0 = x^2 - 5x + 6 \Rightarrow 0 = (x-2)(x-3) \Rightarrow x = 2,\ x = 3$ or $(2, 0), (3, 0)$

10. y-intercept: Substitute $x = 0$: $|0| - y = 2 \Rightarrow -y = 2 \Rightarrow y = -2$ or $(0, -2)$
x-intercept: Substitute $y = 0$: $|x| - (0) = 2 \Rightarrow |x| = 2 \Rightarrow x = \pm 2$ or $(-2, 0)$, $(2, 0)$

11. y-intercept: Substitute $x = 0$: $(0)y - 2y = 1 \Rightarrow -2y = 1 \Rightarrow y = -\frac{1}{2}$ or $(0, -\frac{1}{2})$
x-intercept: Substitute $y = 0$: $x(0) - 2(0) = 1 \Rightarrow 0 = 1 \Rightarrow$ no x-intercepts

12. y-intercept: Substitute $x = 0$: $(0)^2 + 4(0) - 2(0)y = y + 5 \Rightarrow 0 = y + 5 \Rightarrow y = -5$ or $(0, -5)$
x-intercept: Substitute $y = 0$: $x^2 - 4x - 2x(0) = 0 + 5 \Rightarrow x^2 - 4x = 5 \Rightarrow$
$x^2 - 4x - 5 = 0 \Rightarrow (x-5)(x+1) = 0 \Rightarrow x = 5,\ x = -1$ or $(-5, 0), (1, 0)$

13. y-intercept: Substitute $x = 0$: $y = \frac{1}{0} \Rightarrow$ no y-intercepts
x-intercept: Substitute $y = 0$: $0 = \dfrac{1}{x} \Rightarrow 0 = 1 \Rightarrow$ no x-intercepts

14. a) x-axis: Substitute $-y$ for y: $2x - (-y) = 0 \Rightarrow 2x + y = 0 \Rightarrow$ No symmetry
y-axis: Substitute $-x$ for x: $2(-x) - y = 0 \Rightarrow -2x - y = 0 \Rightarrow$ No symmetry
Origin: Substitute $-y$ for y and $-x$ for x: $2(-x) - (-y) = 0\ -2x + y = 0$
Multiplying both sides by -1: $2x - y = 0 \Rightarrow$ Symmetry
Conclusion: Symmetry about the origin only.
b) x-axis: Substitute $-y$ for y: $2x^2 - (-y) = 0 \Rightarrow 2x^2 + y = 0 \Rightarrow$ No symmetry
y-axis: Substitute $-x$ for x: $2(-x)^2 - y = 0 \Rightarrow 2x^2 - y = 0 \Rightarrow$ Symmetry
Origin: Substitute $-y$ for y and $-x$ for x: $2(-x)^2 - (-y) = 0\ 2x^2 + y = 0 \Rightarrow$ No symmetry
Conclusion: Symmetry about the y-axis only.
c) x-axis: Substitute $-y$ for y: $-y = x^3 - 9x \Rightarrow y = -x^3 + 9x \Rightarrow$ No symmetry
y-axis: Substitute $-x$ for x: $y = (-x)^3 - 9(-x) \Rightarrow y = -x^3 + 9x \Rightarrow$ No symmetry
Origin: Substitute $-y$ for y and $-x$ for x: $(-y) = (-x)^3 - 9(-x) \Rightarrow -y = -x^3 + 9x$

Multiplying both sides by -1: $y = x^3 - 9x \Rightarrow$ Symmetry

Conclusion: Symmetry about the origin only.

d) x-axis: Substitute $-y$ for y: $x^2 + (-y)^2 - 6x = 2 \Rightarrow x^2 + y^2 - 6x = 2 \Rightarrow$ Symmetry

y-axis: Substitute $-x$ for x: $(-x)^2 + y^2 - 6(-x) = 2 \Rightarrow x^2 + y^2 + 6x = 2 \Rightarrow$ No symmetry

Origin: Substitute $-y$ for y and $-x$ for x: $(-x)^2 + (-y)^2 - 6(-x) = 2 \Rightarrow$
$x^2 + y^2 + 6x = 2 \Rightarrow$ No symmetry

Conclusion: Symmetry about the x-axis only.

e) x-axis: Substitute $-y$ for y: $x(-y) = x|x^2 - 1| \Rightarrow -xy = x|x^2 - 1| \Rightarrow$ No symmetry

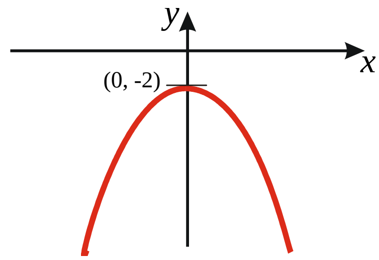
y-axis: Substitute $-x$ for x: $(-x)y = (-x)\left|(-x)^2 - 1\right| \Rightarrow -xy = -x|x^2 - 1| \Rightarrow$
$xy = x|x^2 - 1| \Rightarrow$ Symmetry

Origin: Substitute $-y$ for y and $-x$ for x: $(-x)(-y) = (-x)\left|(-x)^2 - 1\right| \Rightarrow$
$xy = -x|x^2 - 1| \Rightarrow$ No symmetry

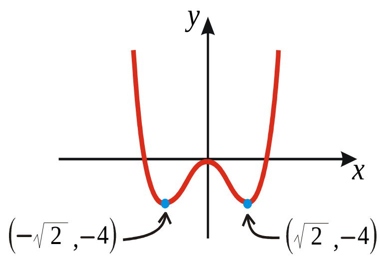
Conclusion: Symmetry about the y-axis only.

f) Symmetry about the x-axis, y-axis, and origin. You prove it!

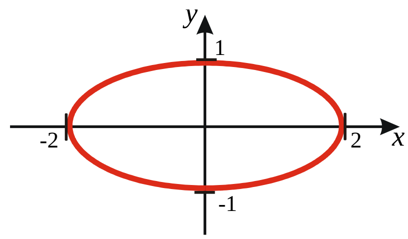
15. a) symmetry about the y-axis only



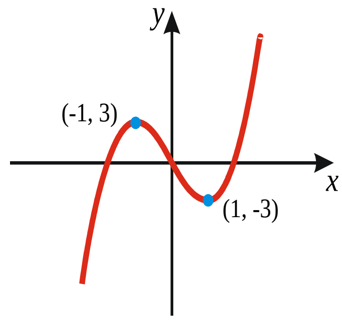
b) symmetry about the y-axis



c) Symmetry about the x-axis, y-axis, and origin.



d) Symmetry about the origin.



16. If $x^2y - 5y^3 = x^4$, then we have

$$x^4 - yx^2 + 5y^3 = 0$$

$$x^2 = \frac{y \pm \sqrt{y^2 - 20y^3}}{2}$$

$$= \frac{y \pm |y|\sqrt{1 - 20y}}{2}$$

Setting $y = \frac{1}{20}$ we have

$$x^2 = \frac{1/20}{2}$$

$$= \frac{1}{40}$$

Thus, $x = \pm 1/\sqrt{40} \approx \pm 0.158$

Chapter 3C - Linear Equations in Two Variables

Linear Equations in Two Variables

In this section we will examine equations in x and y where both variables are raised to the first power.

Definition: Any equation that can be written in the form $Ax + By = C$, where A and B are not both 0, is called a **linear equation**.

Question: According to the definition, which of the following are linear equations?

a) $2x + y = 3$ b) $x - 3 = 0$ c) $y = -3x + 1$ d) $x^2 + y = 1$ e) $2x + y^2 = 5$

Answer:

a) Yes. $A = 2,\ B = 1,\ C = 3.$ x and y are raised to the first power.

b) Yes. $x - 3 = 0$ can be written as $x + 0y = 3$, so that $A = 1,\ B = 0,\ C = 3.$ A and B are not both 0, the variables are raised to the first power.

c) Yes. $y = -3x + 1$ can be written as $3x + y = 1.$ $A = 3,\ B = 1,\ C = 1.$ Both variables are raised to the first power.

d) No. x is raised to the second power.

e) No. y is raised to the second power.

Linear equations can be graphed as any other equation by plotting points.

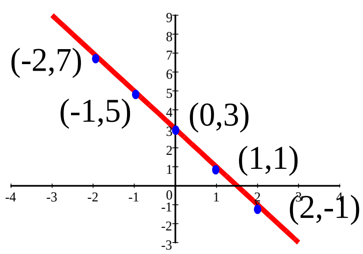
Example 1: Graph the linear equation $2x + y = 3.$

Solution: Solve for y. $y = -2x + 3$

Fill in a table of values

x	y	(x,y)
-2	7	$(-2,7)$
-1	5	$(-1,5)$
0	3	$(0,3)$
1	1	$(1,1)$
2	-1	$(2,-1)$

Plot the points and sketch.



Notice that all of the points were on a line. You might guess that this is why these equations are called linear equations.

The graph of any linear equation is a straight line. Because any two points determine a line, we can graph a linear equation by plotting at least two points. The intercepts are a good choice because they are usually easy points to find.

Question: What are the x- and y-intercepts of the above graph?

Answer:

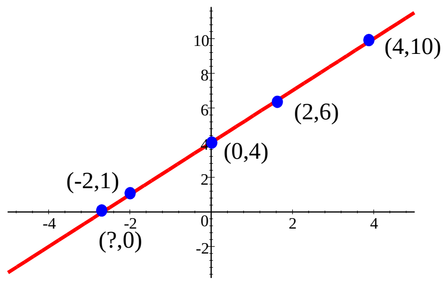
- **y**-intercept: Substitute $x = 0.$ $2(0) + y = 3 \Rightarrow y = 3$ or $(0,3)$
- **x**-intercept: Substitute $y = 0.$ $2x + 0 = 3 \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$ or $\left(\frac{3}{2},0\right)$

Example 2: Graph the linear equation $3x - 2y = -8$.

Solution: Solve for y: $-2y = -3x - 8 \Rightarrow y = \frac{3}{2}x + 4$

Make a table of values: Plot the points and draw the line:

x	y	(x,y)
-2	1	$(-2,1)$
0	4	$(0,4)$
2	7	$(2,7)$
4	10	$(4,10)$



We can see from both the table and the graph that the y-intercept is 4.

Question: What is the x-intercept?

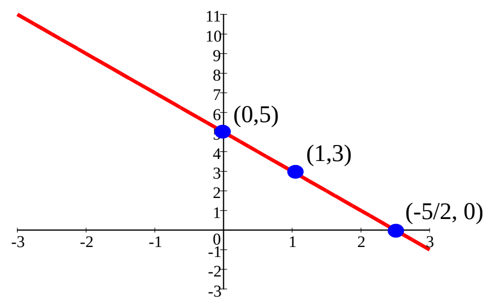
Answer: If $y = 0$, $3x - 2(0) = -8 \Rightarrow 3x = -8 \Rightarrow x = -\frac{8}{3}$. The x-intercept is $-\frac{8}{3}$.

Example 3: Graph the linear equation $y = -2x + 5$.

Solution: We recognize that $y = -2x + 5$ is a linear equation.

Make a table of values: Plot the points and draw the line:

x	y	(x,y)
0	5	$(0,5)$
$-\frac{5}{2}$	0	$\left(-\frac{5}{2},0\right)$
2	1	$(2,1)$



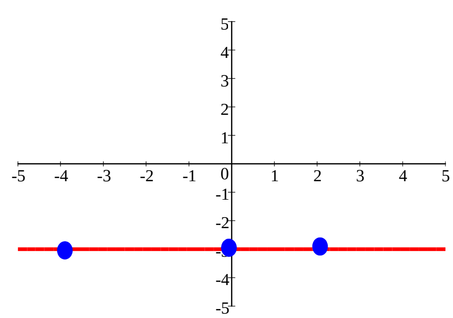
Example 4: Graph the linear equation $2y + 6 = 0$.

Solution: We recognize that $2y + 6 = 0$ is a linear equation. Solving for y: $y = -3$.

Note that the equation does not contain a variable x. The only condition that exists for a point to be included as a solution to the equation is that y must be -3.

Make a table of values: Plot the points and draw the line:

x	y	(x,y)
0	-3	$(0,-3)$
-5	-3	$(-5,-3)$
2	-3	$(2,-3)$



All points with y-coordinate -3 lie on the horizontal line 3 units below the x-axis.

In the following investigation you may use a graphing calculator **but record the table of values and the graph of each equation on graph paper before graphing the next equation**. **As you graph each equation**, **observe the results before graphing the next equation**.

- If you have a TI-82 or TI-83, you can enter the equation using the "Y=" button on the top left of the display. (You must solve the equation for y first.)
- Set the viewing window using the "WINDOW" to the right of the "Y=" button.
- The following viewing window gives a FRIENDLY WINDOW: xmin=–4.7, xmax=4.7. xscl tells you how many units you want each tick mark to represent. ymin=–3.1, ymax=3.1 will give a SQUARE WINDOW.
- Otherwise, the range values can be manipulated to include the part of the graph you want to see. You can multiply each of the above settings to get other friendly and square windows.
- If you do not use a friendly window, you will get large decimals when you trace the graph using the "TRACE" button. Use right and left arrows to move the cursor over the graph. The "x=" and"y=" at the bottom of the screen will give the coordinates of the point highlighted by the cursor. The cursor may skip over the integral x-intercepts if the window is not a friendly window.

Example 5: Graph the following lines on the same coordinate system.

$y_1 = x+1$ $\qquad y_2 = 3x+1$ $\qquad y_3 = \frac{1}{2}x+1$

a) Which number was different in each equation? What effect did this have on the graph?

b) Which number was the same in each equation? What did each line have in common?

Solution:

a) The coefficient of x was different in each equation. The coefficient of x affected the steepness of the graph.

b) The constant was 1 in each equation. They all had a y-intercept of 1.

Example 6: Graph the following lines on the same coordinate system.

$y_1 = x$ $\qquad y_2 = x+2$ $\qquad y_3 = x-1$

a) Which number was different in each equation? What effect did this have on the graph?

b) What number was the same in each equation? What did each line have in common?

Solution:

a) The constant was different in each equation. The y-intercepts were different in each line.

b) The coefficient of x was 1 in each equation. Each line had the same steepness.

Example 7: Graph the following lines on the same coordinate system and compare with the lines in Example 5. What is different between the equation of the lines in this example and the lines in Example 5? What effect did this difference have on the graphs?

$y_1 = -x+1$ $\qquad y_2 = -3x+1$ $\qquad y_3 = -\frac{1}{2}x+1$

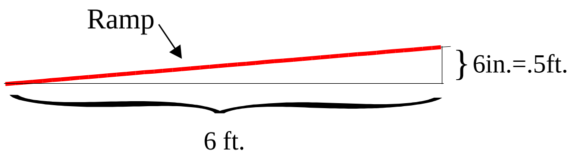
Solution: The coefficients of x in Example 5 are positive, but they are negative in this example. Graphs in Example 5 go uphill from left to right. Graphs in this example go downhill from left to right.

Conclusions: (Graphs of lines in the form $y = mx+b$.)

- m, the coefficient of x, affects the steepness of the line. The larger $|m|$, the steeper the line.
  When $m > 0$, the line goes uphill from left to right. We say the line is increasing.
  When $m < 0$, the line goes downhill from left to right. We say the line is decreasing.
- The number b is the y-intercept. If $x = 0$, then $y = m(0)+b \Rightarrow y = b$

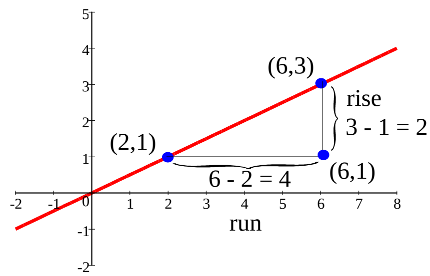
Slopes of Lines

The idea of steepness has applications in many areas of our lives. When we build a house, we must determine the steepness of a roof called the "pitch of the roof". Certainly if we are skiers or cyclists, we are interested in the steepness of a ski slope or roadway. In some cases it is important that we be able to measure steepness. For example, the Americans with Disabilities Act requires that a wheel chair ramp rise no more than 6 inches for 6 feet of horizontal distance, as shown below .



In the above example, we would call the vertical change the rise and the horizontal distance the run. We describe the steepness as the ratio of the rise to the run.

Consider the line below that goes through the points $A(2,1)$ and $B(6,3)$.



The rise can be calculated by subtracting the *y*-coordinates of the given points and the run can be calculated by subtracting the *x*-coordinates of the points. The ratio of the rise to the run $\frac{2}{4}$ can be simplified to $\frac{1}{2}$. This ratio is called the slope of the line.

Definition: The **slope** of a line, *m*, is the ratio of the change in *y* to the change in *x*.

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y\ (\Delta y)}{\text{change in } x\ \ (\Delta x)}$$

The following formula generalizes the process we used above to calculate the slope of any line if we are given two points on the line: $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$. The rise (Δy) and the run (Δx) can be calculated as follows: $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$.

Definition: The **slope** of the line through points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example 1: Plot each pair of points on graph paper and find the slope of the line:

a) $(-3,5)$ and $(4,-2)$ b) $(5,7)$ and $(-1,7)$ c) $(-2,-3)$ and $(5,6)$ d) $(-2,5)$ and $(-2,-1)$

Solution:

a) $m = \frac{-2-5}{4-(-3)} = \frac{-7}{7} = -1$

b) $m = \frac{7-7}{-1-(5)} = \frac{0}{-6} = 0$

c) $m = \frac{6-(-3)}{5-(-2)} = \frac{9}{7}$

d) $m = \frac{-1-5}{-2-(-2)} = \frac{-6}{0} = \textit{undefined}$

Example 2: Use your results to complete each sentence.

a) The line goes uphill from left to right when the slope is ___.

b) The line goes downhill from left to right when the slope is __.

c) The line is horizontal when the slope is ____.

d) The line is vertical when the slope is ____.

Answers: a) positive b) negative c) 0 d) undefined

Investigation of Slopes with a Grapher

Enter the equation $y = 3x + 1$ **into your grapher. Use TABLE mode to answer the following questions**.

Question: How much does y change every time x increases by 1 unit?

Answer: 3 units. For example when x changes from 0 to 1, y changes from 1 to 4.

Quesiton: How much does y change every time x increases by 2 units?

Answer: y changes 6 units. For example, when x changes from 0 to 2, y changes from 1 to 7, a total of 6 units.

Question: How much does y change every time x increases by 3 units?

Answer: 9 units. For example, when x changes from 2 to 5, y changes from 7 to 16, a total of 9 units.

Question: Write each of the above answers as a ratio: $\dfrac{\text{change in } y}{\text{change in } x} = \dfrac{\triangle y}{\triangle x}$. Are these ratios equivalent?

Answer: $\frac{3}{1} = \frac{6}{2} = \frac{9}{3}$. All these ratios are equivalent to $\frac{3}{1}$.

Quesiton: Compare the slope of the line and the coefficient of x in the equation of the line.

Answer: The slope of the line is $\frac{3}{1} = 3$. The coefficient of x in the equation of the line is 3.

Example 3: Enter the equation $y = -\frac{2}{3}x + 4$ into your grapher.

a) Use two points from your table to calculate the slope of the line. Compare with the coefficient of x in the equation.

b) Try several more examples of lines in the form $y = mx + b$. What can you conclude about the relationship between the coefficient of x of a linear equation that has been solved for y and the slope of the line?

Solution:

a) $m = -\frac{2}{3}$ which is also the coefficient of x in the equation.

b) The coefficient of x is the slope of the line.

Conclusions:

- The coefficient of x is the slope of the line.
- If the slope is positive, the graph goes uphill from left to right. We say the line is increasing.
- If the slope is negative, the graph goes downhill from left to right. We say the line is decreasing.

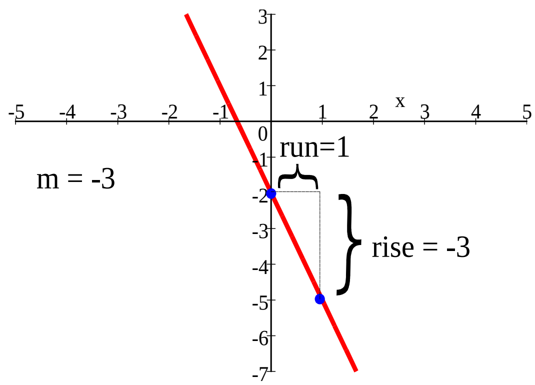
Not only can we find the slope of a line by solving its equation for y, but the form $y = mx + b$ is especially handy in graphing lines. We must use the "$y =$" form to enter the equation into the grapher, but it also makes pencil and paper graphing easier, as well.

To graph a line using the $y = mx + b$ form,

- Since the y-intercept is b, plot the point $(0, b)$ on the y-axis.
- From the y-intercept, apply the slope, $m = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{\triangle y}{\triangle x} = \dfrac{rise}{run}$ to locate a second point on the line.
- Draw the line.

Example 4: Graph $y = -3x - 2$.

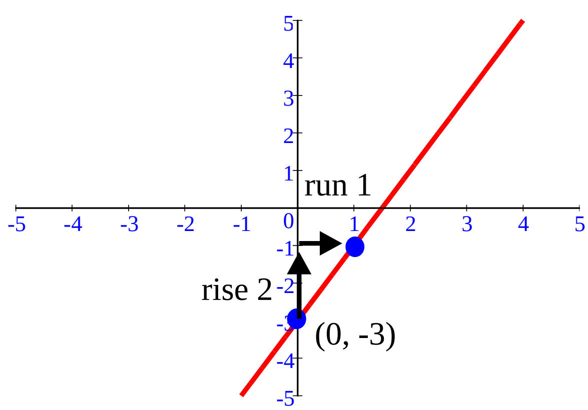
Solution: The y-intercept is -2, so we plot the point $(0, -2)$. The slope is $-3 = \frac{-3}{1}$ which means that y decreases 3 units when x increases 1 unit. We then count one unit to the right and 3 units down from the y-intercept to plot our second point, and draw the line.



Since the slope of the line is negative, we expect the line to go downhill from left to right. We say this line is decreasing.

Example 5: Find the slope and y-intercept of the line $4x - 2y = 6$. Then graph.

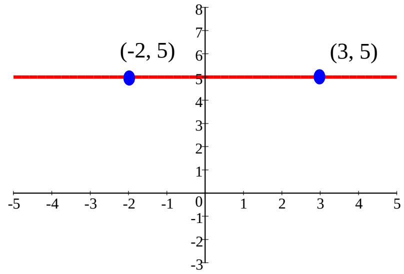
Solution: We must first solve the equation for y: $4x - 2y = 6 \quad \Rightarrow \quad -2y = -4x + 6 \quad \Rightarrow$ $y = 2x - 3$. Therefore, the slope $m = 2$ and the y-intercept $b = -3$. We first plot the point $(0, -3)$. From that point we use the slope $\frac{2}{1}$ to rise 2 and run 1. Plot the second point and draw the line.



We expect the line to go uphill from left to right since the slope is positive. We say this line is increasing.

Horizontal and Vertical Lines

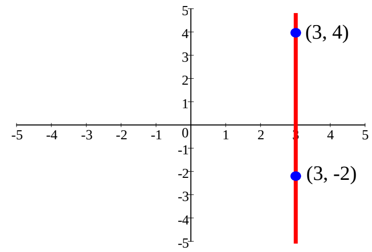
Consider the horizontal line below:



Using the points to calculate the slope gives $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-5}{-2-3} = \frac{0}{-5} = 0.$

Because the y-coordinates are equal on any horizontal line, $y_2 - y_1 = 0$, the slope of a horizontal line is 0.

In a similar manner we can determine the slope of any vertical line.



Calculating its slope gives $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-2)}{3-3} = \frac{5}{0} =$ undefined.

Because the x-coordinates are equal on any vertical line, $x_2 - x_1 = 0$. The slope of a vertical line is undefined, or we say that a vertical line has no slope.

NOTE: The slope of a vertical line is NOT 0. Consider skiing along a horizontal line as compared to skiing down a vertical line. There is a HUGE difference!!!

Parallel and Perpendicular Lines

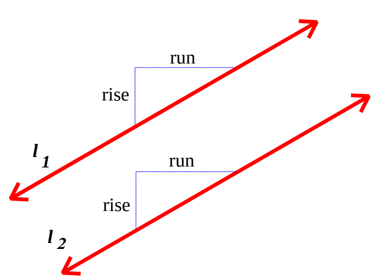
Relationships between pairs of parallel and perpendicular lines and their slopes exist and prove useful in a variety of mathematical settings.

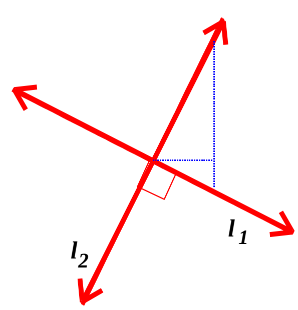
Parallel Lines:

Equal slopes

Perpendicular Lines:

Slopes are negative reciprocals





Example 6: Investigate the following pairs of lines. If you use a calculator, use a square viewing window. (For TI-82 or TI-83, use the screen: $[-4.7, 4.7]$ by $[-3.1, 3.1]$.)

Draw the graphs of each pair of lines on a single set of axes and record the slope of each line in the pair.

a) $y = \frac{1}{2}x - 5$ b) $y = -3x + 4$ c) $y = \frac{2}{3}x + 5$ d) $y = -4x + 2$

$y = \frac{1}{2}x + 4$ $y = -3x + 9$ $y = -\frac{3}{2}x + 1$ $y = \frac{1}{4}x$

Question: What is your conclusion about lines whose slopes are equal?
Answer: Lines whose slopes are equal **appear** to be parallel.

Question: What is your conclusion about lines that are perpendicular?
Answer: Lines that have slopes that are negative reciprocals **appear** to be perpendicular.

Example 7: Find the slope of any line that is a) parallel to ($\parallel$) and b) perpendicular to ($\perp$) the following

1. the line through the points $A(3, -2)$ and $B(-5, 8)$.
2. the line $3x - 5y = 10$.

Solution:

1. $m_{AB} = \dfrac{8-(-2)}{-5-3} = -\dfrac{5}{4}$

a) slope of line parallel to AB: $m_{\parallel} = -\dfrac{5}{4}$ (same slope)

b) slope of line perpendicular to AB: $m_{\perp} = \dfrac{4}{5}$ (negative reciprocal)

2. Slope of line $3x - 5y = 10$:

$-5y = -3x + 1 \quad \Rightarrow \quad y = \dfrac{3}{5}x - \dfrac{1}{5} \quad \Rightarrow \quad m = \dfrac{3}{5}$.

a) $m_{\parallel} = \dfrac{3}{5}$.

b) $m_{\perp} = -\dfrac{5}{3}$

Example 8: If the line through $(3, 6)$ and $(2, y_2)$ is parallel to $3x - 4y = 8$, find the value for y_2.

Solution: Find the slope of the given line by solving for y:

$-4y = -3x + 8 \Rightarrow y = \dfrac{3}{4}x - 2$. The slope of the given line is $m = \dfrac{3}{4}$.

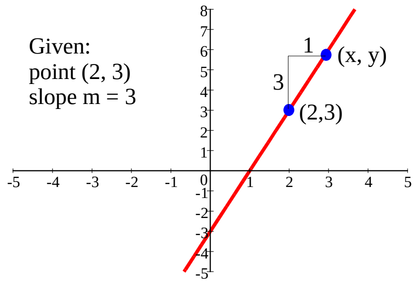
Since parallel lines have equal slopes, the slope of the line through $(3, 6)$ and $(2, y_2)$ must be $\frac{3}{4}$. Using the slope formula, we get

$$\frac{y_2 - 6}{2-3} = \frac{3}{4}$$
$$4(y_2 - 6) = 3(-1)$$
$$4y_2 - 24 = -3$$
$$4y_2 = 21$$
$$\text{Therefore, } y_2 = \frac{21}{4}$$

Equations of Lines

Previously, we saw that the graph of any equation that can be written in the form $Ax + By = C$, where A, B are not both 0 is a straight line. The equation $Ax + By = C$ is called the general form of the equation of a straight line. Moreover, we were able to write a linear equation as $y = mx + b$ by solving the equation for y. This form is called the slope-intercept form.

In this section, we will write an equation for a line given information about the line. Consider the following:



We can find an equation for the line that passes through the point $(2,3)$ with slope $m = 3$ by using the slope formula. Let (x,y) be any point on the line except the given point $(2,3)$. Then

$$3 = \frac{y-3}{x-2} \Rightarrow$$

$$3(x-2) = y-3$$

Simplifying the equation, we get

$$3x - 6 = y - 3$$

$$3x - y = 3$$

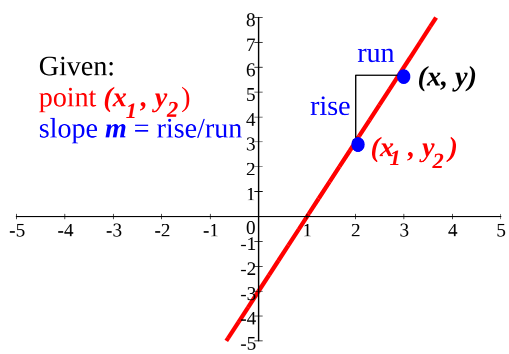
which is the general equation of the line.

Generalizing the process we used to write an equation for a line gives us a formula we can use to find the equation for any line if we know its slope m and a particular point on the line (x_1,y_1). This form is called the point-slope form of an equation of a line.

POINT-SLOPE FORM:

The equation of a line with slope m that passes through point (x_1,y_1) is

$$y - y_1 = m(x - x_1).$$



If (x,y) is any other point on the line, we can use the slope formula to get

$$m = \frac{y-y_1}{x-x_1} \Rightarrow y - y_1 = m(x - x_1)$$

Example 1: Write an equation for the line with slope $\frac{1}{2}$ that passes through $(2,-1)$.

Solution: Since we are given a point on the line and its slope we can use the point-slope form to write the equation.

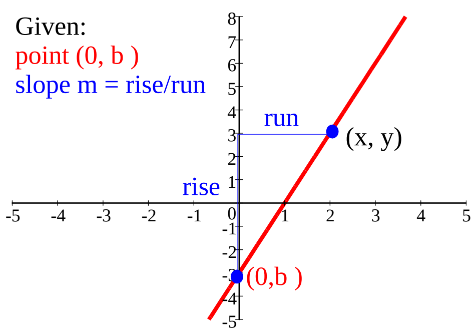
$$y - y_1 = m(x - x_1) \quad \text{Substituting, we get}$$
$$y - (-1) = \frac{1}{2}(x - 2) \quad \text{Multiplying by 2}$$
$$2(y + 1) = x - 2$$
$$x - 2y = 4$$

We can verify what we learned earlier about a line in the form $y = mx + b$. If we know the slope m and the y-intercept b, then we can use the slope-intercept form to write the equation of the line.

SLOPE-INTERCEPT FORM:

The equation of a line with slope m and y-intercept b is

$$y = mx + b.$$



If we know the y-intercept, b, then the point $(0, b)$ is a point on the line. Substituting into the point-slope form, we get

$$y - y_1 = m(x - x_1)$$
$$y - b = m(x - 0) \;\Rightarrow\; y - b = mx \;\Rightarrow\; y = mx + b.$$

Example 2: Write an equation for the line with slope 5 and y-intercept 2.

Solution: Since we are given the slope 5 and y-intercept, 2, we can substitute in the slope-intercept form. The equation of the line is $y = 5x + 2$.

The slope-intercept form is the easiest to use, so be on the lookout for points that have an x-coordinate of 0.

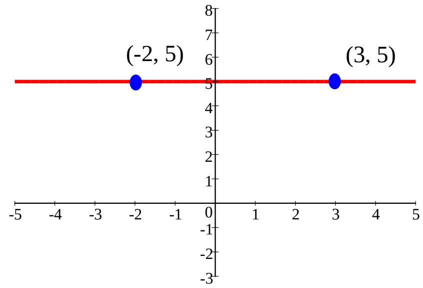
Example 3: Write the equation of the line with slope -1 through the point $(0, 4)$.

Solution: Although the equation can be found using the point-slope form, the point $(0, 4)$ is on the y-axis. Therefore, the equation of the line can be written more easily by substituting into the slope-intercept form. Given $m = -1$ and $b = 4$, the equation is $y = -x + 4$.

Lines that have been written using point-slope can be simplified to slope-intercept form which is often preferred over the general form.

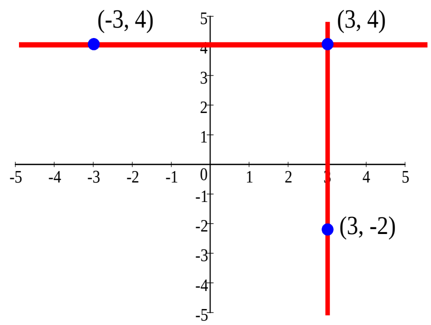
Horizontal Lines

For two points to line up horizontally, they must have the same y-value. Note that all the points on the horizontal line below will have a y-coordinate of 5. Therefore, the equation $y = 5$ describes the set of points graphed below.



The equation of a horizontal line through the point (a, b) is $y = b$.

Example 4: What is the equation of the horizontal line through $(3, 4)$ shown below?



Answer: $y = 4$.

Vertical Lines

Similarly, points must have the same x-coordinate to line up vertically.
The equation of a vertical line through the point (a, b) is $x = a$.

Example 5: What is the equation of the vertical line shown above?
Answer: $x = 3$.

Summary:

- Given slope m and point (x_0, y_0), use **point-slope form**: $y - y_0 = m(x - x_0)$
- Given slope m and y-intercept b: use **slope-intercept form**: $y = mx + b$
- **horizontal line** through (a, b): $y = b$
- **vertical line** through (a, b): $x = a$

Exercises for Chapter 3C - Linear Equations in Two Variables

1. Graph the equation $3x - 5y = 15$

2. Graph the equation $y = 2x - 3$

3. Graph the equation $y = -\frac{1}{3}x + 2$

4. Graph the equation $y - 3 = 0$

5. Graph the equation $x - 2y = 6$

6. Graph the equation $x = -2$.

7. Graph the equation $x - y = 4$

8. Graph the equation $y = 2$

9. Graph the equation $3x + 2y = 10$.

10. Graph the equation $y = -x + 1$

11. Recall that the Americans with Disabilities Act (ADA) requires that a wheel chair ramp rise no more than 6 inches for 6 feet of horizontal distance. What is the slope of a wheel chair ramp?

12. Using the ADA requirements above, complete the table with the rise allowable under the law:

Horizontal Distance Covered (in feet)	**Vertical Distance Allowable (in feet)**
6	
12	
18	
24	
32	
40	

13. Find the slope of the line through each pair of points.

a) $A(-2,-1)$ and $B(3,5)$

b) $C(0,-3)$ and $D(-4,1)$

c) $E(-4,2)$ and $F(2,-3)$

d) $G(\frac{1}{2},\frac{-2}{3})$ and $H(\frac{3}{2},\frac{-5}{3})$

e) $J(-3.3,\ 4.9)$ and $K(1.2,\ -7.5)$

f) $L(3,5)$ and $M(3,-5)$
g) $N(0,-2)$ and $P(3,-2)$

14. Determine whether each line in the previous exercise will go uphill (from left to right), downhill (from left to right), be horizontal, or vertical.

15. Find the slope of any line that is parallel to each of the lines in exercise number 13.

16. Find the slope of any line that is perpendicular to each of the lines in exercise number 13.

17. If the line through $(1,5)$ and $(6,y_2)$ is parallel to the line $3x - 5y = 10$, find y_2.

18. Use slopes to prove that the triangle with vertices $A(-3,-2)$, $B(-2,2)$, and $C(6,0)$ is a right triangle.

19. Find the slope of each of the following lines and determine whether the line will be increasing, decreasing, horizontal or vertical.
a) $y = -x + 2$
b) $y = \frac{2}{3}x - 1$
c) $2x - y = 0$
d) $3x - 2y = 8$
e) $2y - 8 = 0$
f) $2x = -2$

20. Graph each of the lines in exercise 19*a* and 19*b* using the slope and *y*-intercept.

21. Find the slope of each of the following and graph both lines on the same axes.
a) line through $(0,3)$ that is perpendicular to $y = -\frac{3}{4}x + 1$
b) line through $(0,-1)$ that is parallel to $3x + y = 2$
c) line through $(0,0)$ that is parallel to $2x - y = 5$
d) line through $(0,2)$ that is perpendicular to $2x - y = 5$

22. Write an equation for each of the following lines. Write the answer in i) slope-intercept form and ii)general form.
a) through $(2,3)$ with slope -1.
b) through $(5,-1)$ with slope $\frac{2}{5}$.
c) through $(0,3)$ with slope $-\frac{3}{4}$.
d) through $(6,2)$ and $(7,-1)$.
e) through $(3,-2)$ and $(5,-2)$
f) through $(5,-2)$ and $(0,6)$
g) through $(4,-2)$ and $(4,5)$
h) vertical line through $(5,9)$
i) horizontal line through $(5,9)$

23. Write an equation for each line. Leave answer in slope-intercept form.

a) line through $(0,3)$ that is perpendicular to $y = -\frac{3}{4}x + 1$

b) line through $(0,-1)$ that is parallel to $3x + y = 2$

c) line through $(4,0)$ that is parallel to $2x - y = 5$

d) line through $(5,2)$ that is perpendicular to $2x - y = 5$

24. If a designer needed a wheelchair ramp to accommodate a 10 foot vertical change in elevation, how much horizontal distance would be required?

25. Explain in your own words what is meant by ”slope”.

26. Graph each of the following equations. Label the axes, including scales for each axis. What is the slope of the linear equation and y-intercept of each and what do they mean in the particular situation?

a) Grandma Jones makes quilts and sells them at the Flea Market. She figured that without including labor her total cost y for making x quilts and renting a $25-booth at the flea market was $y = 20x + 25$.

b) HRL, inc. bought an apartment complex for $240,000. It is depreciating at a rate of $20,000 per year, so that its value y in x years is given by $y = -20,000x + 240000$. ....

c) Ajax Car Rental rents a mid-sized car for a flat rate of $25 a day plus $0.25 per mile. The total daily car rental y if x miles are driven is $y = .25x + 25$.

d) If computer salesman Sam Dell is paid a base salary of $1000 per month and a commission of 5% of all sales, his monthly salary y for sales of x dollars is $y = .05x + 1000$.

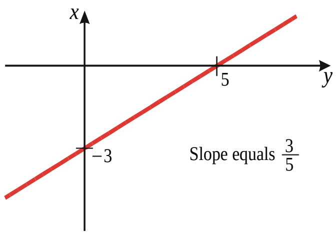
27. TTI pays students $20 per day plus $0.10 per mile to perform a variety of driving tests. Write an equation that models the daily pay for a student. Graph your equation and label the axes.

28. Write an equation to model the monthly cost of producing x tons of widgets if the fixed costs are $3500 per month and the cost per ton of widgets is $200. Graph the equation and label the axes.

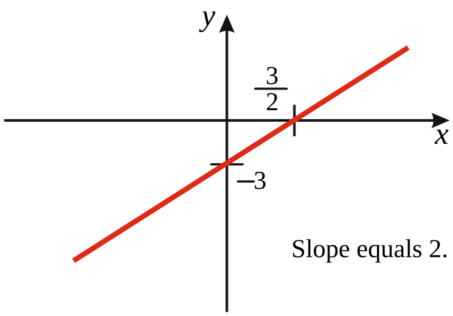
29. Write an equation for the perpendicular bisector of the line segment connecting $A\,(-4,1)$ and $B(6,5)$. The perpendicular bisector of a line segment $\overline{AB}$ is the line that is pendicular to $\overline{AB}$ and cuts $\overline{AB}$ into two equal pieces.

Answers to Exercises for Chapter 3C - Linear Equations in Two Variables

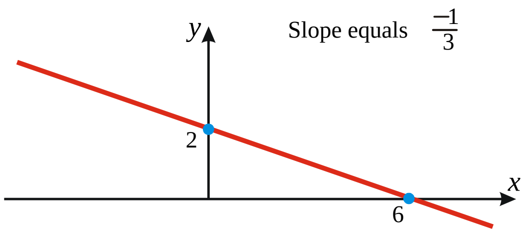
1. Solve for y : $-5y = -3x + 15 \quad \Rightarrow \quad y = \frac{3}{5}x - 3 \Rightarrow m = \frac{3}{5} \quad b = -3$. Plot the y-intercept $(0, -3)$ and apply the slope $rise = 3$ and $run = 5$.



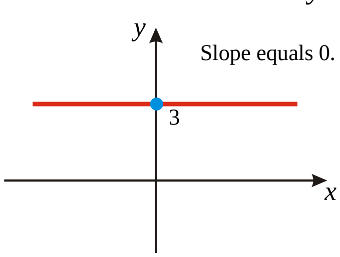
2. Equation is in slope-intercept form: $m = \frac{2}{1} \quad b = -3$. Plot $(0, -3)$ and apply slope: right 1 and up 2



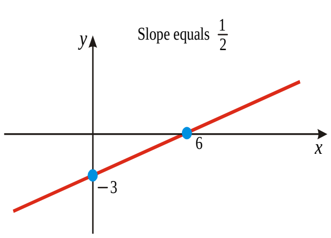
3. Equation is in slope-intercept form: $m = \frac{-1}{3} \quad b = 2$. Plot $(0, 2)$ and apply slope: right 3 and down 1



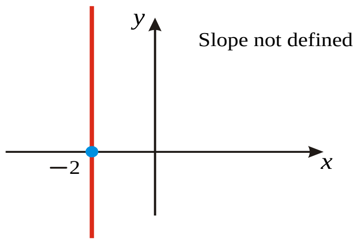
4. Horizontal line: solve for y: $y = 3$.



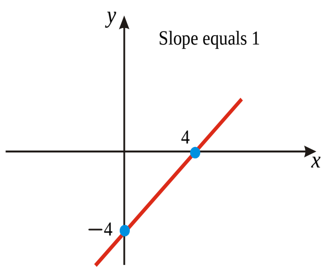
5.



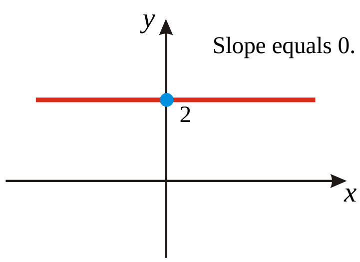
6. Vertical Line 2 units to the left of the *y*-axis.



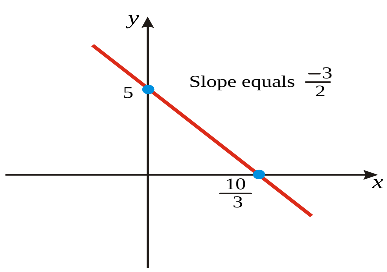
7. solve for y : $y = x - 4$



8. Horizontal line 2 units above the *x*-axis.



9.



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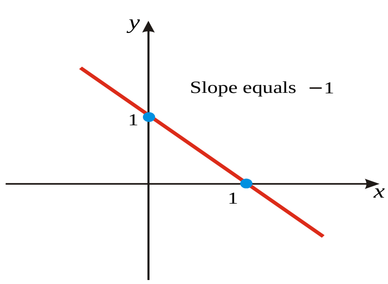


11. Since the slope is a ratio, the rise and the run must be written using the same units: feet or inches. $m = \frac{6\ inches}{72\ incher} = \frac{1}{12}$ or $m = \frac{\frac{1}{2}\ feet}{6\ \ feet} = \frac{1}{12}$. In either case, $m = \frac{1}{12}$

12.

Horizontal Distance Covered (in feet)	Vertical Distance Allowable (in feet)
6	.5
12	1
18	1.5
24	2
32	$2\frac{2}{3}$
40	$3\frac{1}{3}$

13. Find the slope of the line through each pair of points.

a) Using the slope formula, $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{-1-5}{-2-3} = \dfrac{-6}{-5} = \dfrac{6}{5}$

b) Using the slope formula, $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{-3-1}{0-(-4)} = \dfrac{-4}{4} = -1$

c) Using the slope formula, $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{2-(-3)}{-4-(2)} = \dfrac{5}{-6} = -\dfrac{5}{6}$

d) Using the slope formula, $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{\frac{-2}{3} - (\frac{-5}{3})}{\frac{1}{2} - (\frac{3}{2})} = \dfrac{1}{-1} = -1$

e) Using the slope formula, $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{4.9-(-7.5)}{-3.3-1.2} = \dfrac{12.4}{-4.5} = -\dfrac{124}{45}$

f) Using the slope formula, $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{5-(-5)}{3-3} = \dfrac{10}{0} =$ *undefined*. Note that two points with the same *x*-coordinate must be located on a vertical line.

g) Using the slope formula, $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{-2-(-2)}{0-3} = \dfrac{0}{-3} = 0$. Note that two points with the same *y*-coordinate must be located on a horizontal line.

14. a) uphill– slope is positive.
b) downhill–slope is negative.
c) downhill–slope is negative.
d) downhill–slope is negative.
e) downhill–slope is negative.
f) vertical–slope is undefined.
g) horizontal–slope is zero.

15. Slopes of parallel lines are equal, so the slopes should be the same as the original line.

a) $\frac{6}{5}$ b) -1 c) $-\frac{5}{6}$ d) -1 e) $-\frac{124}{45}$ f) undefined g) 0

16. Slopes of perpendicular lines are negative reciprocals:

a) $-\frac{5}{6}$ b) 1 c) $\frac{6}{5}$ d) 1 e) $\frac{45}{124}$ f) 0 g) *undefined*

17. Since the line through $(1,5)$ and $(6,y_2)$ must be parallel to the line $3x - 5y = 10$, the slopes must be equal. Find the slope of $3x - 5y = 10$ by solving for y: $y = \frac{3}{5}x - 2 \Rightarrow m = \frac{3}{5}$.
Express the slope of the line through the two points using the slope formula,
$m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{5 - y_2}{1-6} = \dfrac{5-y_2}{-5}$. Set the two equal to each other and solve for y_2:
$\dfrac{5-y_2}{-5} = \dfrac{3}{5} \Rightarrow 25 - 5y_2 = -15 \Rightarrow y_2 = 8$

18. To show that ΔABC is a right triangle, we must show that it has a right angle. Since right angles are formed be the intersection of perpendicular lines, we will find the slope of each line forming the triangle and determine if any pair of the lines are perpendicular.

$m_{AB} = 4 \qquad m_{BC} = -\dfrac{1}{4}$

Slopes are negative reciprocals $\Rightarrow AB \perp BC \Rightarrow \angle B$ is a right angle. Therefore, ΔABC is a right triangle.

19. a) $m = -1 \Rightarrow$ decreasing

b) $m = \frac{2}{3}$ is positive $\Rightarrow$; increasing

c) $m = 2$ is positive $\Rightarrow$ increasing

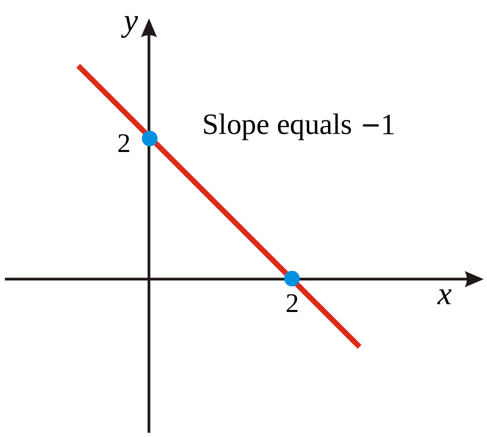
d) $m = \frac{2}{3}$ is positive $\Rightarrow$ increasing

e) $m = 0 \Rightarrow$ horizontal

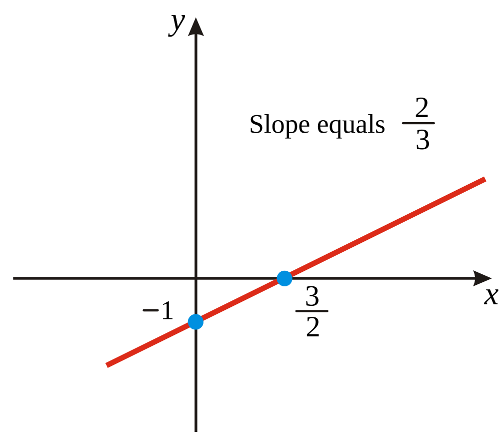
f) $m = \textit{undefined} \Rightarrow$ vertical

20.

a)

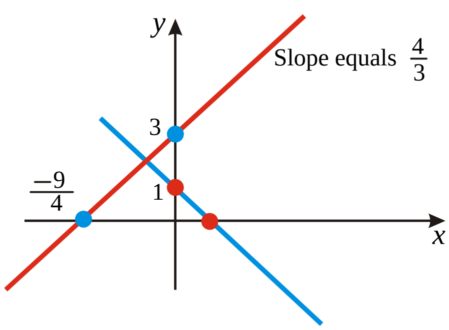


b)

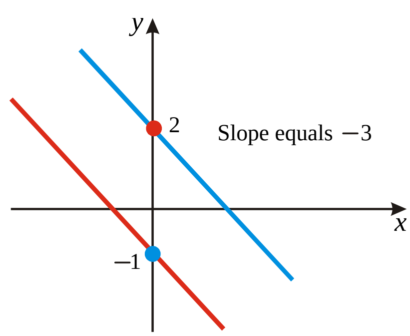


21.

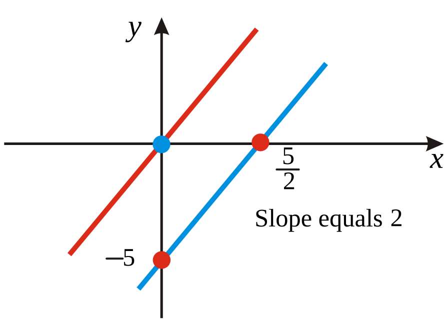
(a)



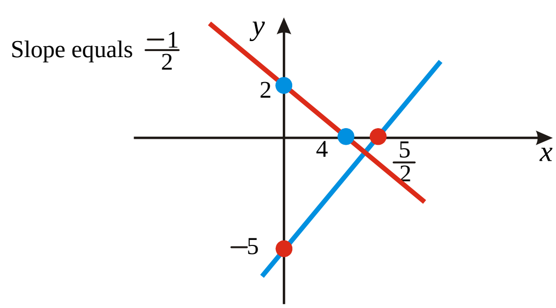
(b)



(c)



(d)



22. a) Given a point and the slope we use point-slope form to write the equation:
$y - y_1 = m(x - x_1) \Rightarrow y - 3 = -1(x - 2) \Rightarrow y - 3 = -x + 2$

i) simplifying to slope-intercept we solve for y : $y = -x + 5$

ii) simplifying to general form we bring all terms to the left side: $x + y - 5 = 0$

b) Use point-slope: $y - (-1) = \frac{2}{5}(x - 5)$. Multiply both sides by 5 to clear fractions:
$5(y + 1) = 5 \cdot \frac{2}{5}(x - 5) \Rightarrow 5y + 5 = 2(x - 5) \Rightarrow 5y + 5 = 2x - 10$

i) solve for y: $y = \frac{2}{5}x - 3$

ii) $2x - 5y - 10 = 0$ (We usually manipulate the equation to get the first term to be positive.)

c)

i) Since $(0, 3)$ is the y-intercept and $m = -\frac{3}{4}$, we can substitute into the slope-intercept form: $y = mx + b$: $y = -\frac{3}{4}x + 3$

ii) We must clear the equation of fractions by multiplying through by 4 : $4y = -3x + 12$. Bringing all terms to the left gives us the equation in general form: $3x + 4y - 12 = 0$

d) We must find the slope using the slope formula: $m = \frac{2+1}{6-7} = -3$. Substituting into point-slope using either of the two given points we get $y - 2 = -3(x - 6) \Rightarrow$
$y - 2 = -3x + 18$

i) Solve for y : $y = 3x - 16$

ii) Bring all terms to the left: $3x - y - 16 = 0$

e) Since both y-coordinates are -2, the line is the horizontal line 2 units below the x-axis.

i) $y = -2$ (Note that the slope is 0 and the y-intercept is -2.)

ii) $y + 2 = 0$

f) Find the slope using the slope formula: $m = \frac{-2-6}{5-0} = -\frac{8}{5}$ $(0, 6) \Rightarrow b = 6$. Substitute into $y = mx + b$:

i) $y = -\frac{8}{5}x + 6$

ii) Clear fractions and bring all terms to the left. $8x + 5y - 30 = 0$

g) Since the x-terms are equal, the line is the vertical line 4 units to the right of the y-axis.

i) There is no slope intercept form of the equation : $x = 4$

ii) $x - 4 = 0$

h) All points on the vertical line through $(5, 9)$ must have an x-intercept of 5. Thus,

i) no slope intercept for for the equation $x = 5$ ii) $x - 5 = 0$

i) Horizontal line must have all points with a y-coordinate of 9. Thus,

i) $y = 9$ ii) $y - 9 = 0$

23. a) Line perpendicular to $y = -\frac{3}{4}x + 1$ must have a slope $\frac{4}{3}$. $(0, 3) \Rightarrow b = 3$. Using slope interecept, we get the equation to be $y = \frac{4}{3}x + 3$.

b) Find slope of the given line by writing in slope intercept form: $y = -3x + 2 \Rightarrow$
$m = -3$ so the slope of our parallel line is the same, -3. $(0, -1) \Rightarrow b = -1$. Using slope-intercept form: $y = -3x - 1$.

c) Find slope of given line: $y = 2x - 5 \Rightarrow m = 2 \Rightarrow m_{\parallel} = 2$. We cannot use slope intercept because $(4, 0)$ is and x-intercept, not a y-intercept, so we must use point-slope.
$y - 0 = 2(x - 4) \Rightarrow y = 2x - 8$

d) Slope of given line: $y = 2x - 5 \Rightarrow m = 2$ so that $m_{\perp} = -\frac{1}{2}$. Using point slope:
$y - 2 = -\frac{1}{2}(x - 5) \Rightarrow y = -\frac{1}{2}x + \frac{5}{2} + 2 \Rightarrow y = -\frac{1}{2}x + \frac{9}{2}$

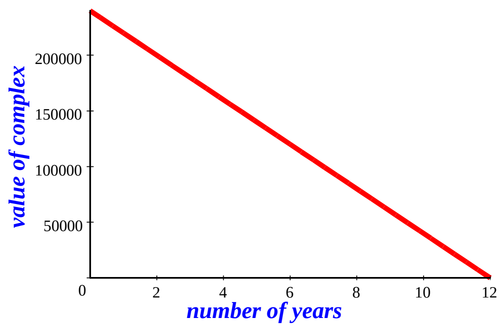
24. 120 feet

25. Your explanation.

26. a)

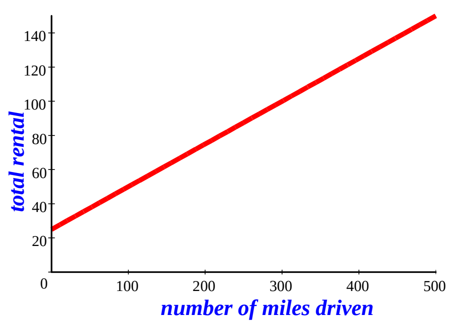
Slope: $m = 20 \Rightarrow$cost/quilt y-intercept: $b = 25 \Rightarrow$rent of booth

b)



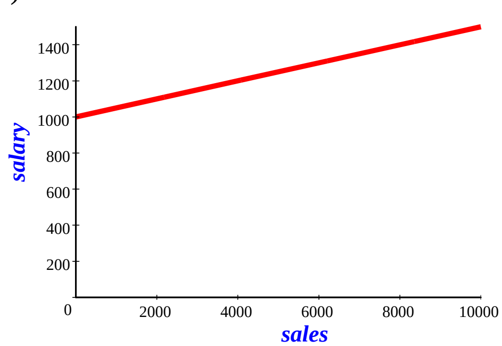
$m = -20000 \Rightarrow$depreciation/year $b = 240,000 \Rightarrow$original value of complex

c)



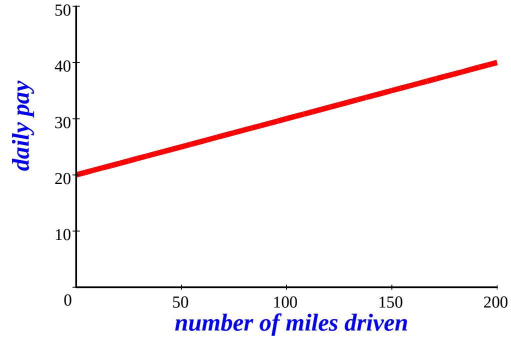
$m = .25 \Rightarrow$rate/mile each day $b = 25 \Rightarrow$flat rate each day

d)

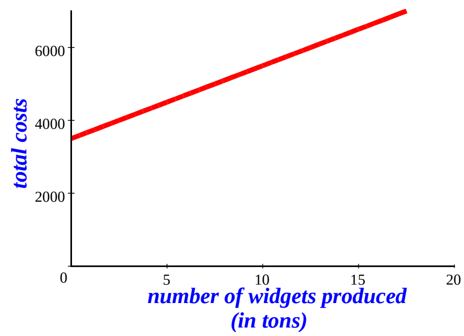


$m = .05 \Rightarrow$commission rate $b = 1000 \Rightarrow$base salary

27. Let x = number of miles driven y = total pay for the day Model: $y = .10x + 20$



28. Let x = number of tons of widgets produced y = total cost of producing x widgets Model: $y = 200x + 3500$



29. $y = -\dfrac{5}{2}x + \dfrac{11}{2}$